Macro, Money, and Finance:
a continuous-time approach

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Course Information

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Classes: Wednesdays 3-6pm. **Location:** Meeting room, Economics Dept. PUC Rio.
Software: Matlab and Python.

Course Description

This course introduces students to modern continuous time modeling techniques at the intersection between macroeconomics, monetary economics and finance. The course has a hands on approach and covers analytic and numerical methods.

This course explores macroeconomic models with financial frictions, emphasizing the role of heterogeneous agents. As opposed to mainstream macro models which emphasize consumption decisions, the course focuses on agents' portfolio choices and how these interact with financial markets. A key feature of the framework is that risk is endogenous, influencing the price of risk and leading to time-varying risk premia. Through this lens, the course examines the implications of financial frictions for asset pricing, macroeconomic dynamics, and policy analysis.

Evaluation

There will be problem sets, a presentation / referee report, and a research proposal. Please, send problem sets to the TA via email. The professor will handle directly presentations and research proposals.

Course Outline

Part 0. Preliminaries

0. **A Brief History of Macroeconomics and Finance.** A helicopter view of how the fields of macroeconomics and finance have evolved and their convergence into continuous-time macro models with financial frictions. Why continuous time modeling?

Part I. Analytic toolbox

1. **Stochastic Calculus Basics.** Measure and probability theory: basic concepts, Standard Brownian Motion, Martingales, Stochastic integral, Ito processes, Ito diffusion, Stochastic Differentiation: Ito formula, Generator of an Ito diffusion, Stochastic optimization: Dynamic Programming, Martingale Representation Theorem, Girsanov Theorem, Kolmogorov Forward Equation, Backward Stochastic Differential Equations (BSDE), Forward-Backward Stochastic Differential Equation (FBSDE), Stochastic Optimization: Stochastic Maximum Principle.

2. **Optimization: Consumption and Portfolio Choice.** We solve a fairly general single-agent consumption-portfolio choice using three approaches: [i] Dynamic Programming (HJB equation), [ii] Stochastic Maximum Principle (the Hamiltonian), [iii] the Martingale Method.

Part II. A baseline model for endogenous risk

3. **A Simple Real Macro Model with Heterogeneous Agents**

   We illustrate how to solve a general equilibrium within this literature using a simple two sector model (Basak Cuoco 1998). This simple model allows for closed form (global) solutions mainly because of (i) the impossibility of less productive agents to manage the productive input (capital) and (ii) logarithmic preferences. Unfortunately, the model lacks some desired properties to study financial crises (e.g., endogenous risk).

   - Basak and Cuoco (1998)

4. **Endogenous Risk Dynamics in Real Macro Model with Heterogeneous Agents**

   We solve a version of the model in Brunnermeier Sannikov (2016), which allows for capital miss-allocation and non-myopic preferences. The model delivers a normal regime around the (stochastic steady state) and a crisis regime when the key sector in the economy (the financial sector) is under-capitalized. The model features endogenous (time-varying) risk and price of risk. Also, it allows to study fire-sales, liquidity spirals, and fat tails. It delivers a volatility paradox and rationalizes how a financial innovation could cause instability. Additional toolboxes: price-taking social planner and change of numeraire.

- Brunnermeier and Sannikov (2016b), Brunnermeier and Sannikov (2014),

5. **Numerical methods for continuous-time models (finite differences)**: Introduction: a general class of equations, examples: valuation equation and HJB, Forward and Backward Equations: HJB and KFE, Finite Difference Schemes: Key Principles, Finite Difference Operator and sign of Matrix M, Explicit Scheme, Implicit Scheme, Stationary Value Function in a Single Step, KFE using Matrix M, General Class of HJB in One Dimension, Solving HJB, Non-monotone Schemes (what can go wrong?), Valuation Equation in m Dimensions, Convex Positive Semidefinite Cone, Some Geometry in Two Dimensions, Algorithm for the 2nd-order Term in Two Dimensions, Assembling M and Solving the Valuation Equation, Solving HJB Equation in m Dimensions.

Part III. Advanced topics

6. **Multiplicity and Stochastic Stability**

   In a simplified version of the model with endogenous risk dynamics, we show that the model allows for equilibria influenced by sunspots or "rational sentiment" (i.e., equilibria that are not Markov in the fundamental states of the economy). We show that rational sentiment helps resolve two puzzling features plaguing models emphasizing balance sheets: [i] financial crises emerge suddenly, featuring large volatility spikes and asset-price declines; [ii] asset price booms, with below-average risk premia, predict busts and financial crises. We introduce basic concepts and a useful lemma of stochastic stability theory.

   - Mendo and Khorrami (2024), Mendo and Khorrami (2025b), Mendo and Khorrami (2025a), Khasminskii (2011).

7. **Epstein Zin preferences and Jumps**

   We generalize the model with endogenous risk dynamics to allow for recursive EZ preferences. Basic stochastic calculus with jumps. We extend the single agent consumption-portfolio problem to include the case where the asset returns are sensitive to jumps. We use jumps in three different contexts: [i] Multiplicity. We extend the information set to include a Poisson shock and show that the model can have equilibria in which equilibrium outcome respond to those shocks. This applies to both, equilibria recursive in fundamentals and those that are not. [ii] Real Poisson shock. We extend the model to add a capital destruction shock that arrives according to a Poisson shock. [iii] Model with financial panics.

   - Li (2021), Mendo (2020)

8. **One sector Monetary Model and Idiosyncratic Risk**

   - Brunnermeier *et al.* (2022b)

- Brunnermeier *et al.* (2022a)

9. **The I Theory of Money with Heterogenous Agents**
   - — and — (2016a). The i theory of money. *Working Paper*

10. **Applications**
    - Silva, D. and Duarte, D. (2023). Machine learning for continuous-time finance, working Paper
    - Maggiori, M. (2017). Financial intermediation, international risk sharing, and reserve currencies. *American Economic Review*, **107** (1), 3038–3071
    - — and Hall, R. (2021). Risk premium shocks can create inefficient recessions. *Review of Economic Studies*, **0**, 1–35
    - — (2020). Risk premia and the real effects of money. *American Economic Review*, **110** (7), 1995–2040
    - Vayanos, D. and Vila, J.-L. (2021). A preferred-habitat model of the term structure of interest rates. *Econometrica*, **89** (1), 77–112
    - Khorrami, P. (2020). The risk of risk-sharing: Diversification and boom-bust cycles. *Working Paper*
    - —, Vandeweyer, Q. and Darracq Paries, M. (2023). Central banking with shadow banks. *Working Paper*
    - — and — (2013). Intermediary asset pricing. *American Economic Review*, **103** (2), 732–770
    - Krishnamurthy, A. and Li, W. (2021). Dissecting mechanisms of financial crises: Intermediation and sentiment. *Working Paper*
    - Di Tella, S. (2017). Uncertainty shocks and balance sheet recessions. *Journal of Political Economy*, **125** (6), 2038–2081
    - He, Z. and Krishnamurthy, A. (2012). A model of capital and crises. *Review of Economic Studies*, **79** (2), 735–777
    - — (2022b). A macro-finance model with realistic crisis dynamics, working Paper
    - — (2021). Why are banks exposed to monetary policy? *American Economic Review*, **110** (7), 1995–2040

11. **More on Numerical Methods**
    - Gopalakrishna, G. (2022a). Aliens and continuous time economies, working Paper

- —, LEE, S. J. and PAPAMICHALIS, T. (2023). Heterogeneous beliefs, risk amplification, and asset returns, working Paper
- FERNÁNDEZ-VILLAVERDE, J., HURTADO, S. and NUÑO, G. (2023). Financial frictions and the wealth distribution. *Econometrica*, **91** (3), 869–901
- D'AVERNAS, A., PETERSEN, D. and VANDEWEYER, Q. (2022). A solution method for continuous-time models, working Paper

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—, — and — (2022b). Safe assets: A dynamic retrading perspective, working Paper.

— and SANNIKOV, Y. (2014). A macroeconomic model with a financial sector. *American Economic Review*, **104** (2), 379–421, Link.

— and — (2016a). The i theory of money. *Working Paper*.

— and — (2016b). Macro, money, and finance: A continuous-time approach. In *Handbook of Macroeconomics*, vol. 2, pp. 1497–1545.

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MENDO, F. (2020). Risky low-volatility environments and the stability paradox. *Working Paper*.

— and KHORRAMI, P. (2024). Rational sentiments and financial frictions. *Working Paper*.

— and — (2025a). Dynamic self-fulfilling fire sales. *Working Paper*.

— and — (2025b). Fear, indeterminacy, and policy responses. *Working Paper*.

SILVA, D. and DUARTE, D. (2023). Machine learning for continuous-time finance, working Paper.

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