Internet Appendix

"Risky low-volatility environments and the stability paradox"
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This appendix presents a generalized version of the model and shows that, as long as financial frictions prevent agents from completely unload the risks of their assets into financial markets, the invariance of capital price and allocation to the exposure to panics (i.e., Theorem 1) is robust to the following features: idiosyncratic productivity shocks, aggregate shocks to the growth rate of capital and to the volatility of idiosyncratic shocks, and a derivative market to trade aggregate risks. Considering heterogeneous Epstein-Zin preferences lead to a (weaker) version of the result: the capital demand of experts relative to households is not affected by the vulnerability to panics.

1 Environment

Consider the environment described in the paper with the following generalizations.

Preferences. Heterogeneous EZ preferences with parameters $(\psi_j, \gamma_j, \rho_j)$ representing the inverse of IES, risk aversion, and discount rate (including the uninsurable death risk).

$$\begin{aligned} U_{j,t} &= \mathbb{E}\left[\int_t^\infty \varphi_j\left(c_{j,s}, U_{j,s}\right) ds\right] \\ \varphi_j(c,U) &= \frac{1}{1-\psi_j}\left\{\rho_j c^{1-\psi_j}\left[(1-\gamma_j)U\right]^{\frac{\psi_j-\gamma_j}{1-\gamma_j}} - \rho_j\left(1-\gamma_j\right)U\right\} \end{aligned} \tag{1}$$

The preferences are symmetric within each group.

Technology. Capital for agent j evolves according to

$$dk_{j,t} = k_{j,t}\left[\left(g_t + \Phi\left(\iota_{j,t}\right) - \delta\right)dt + \sqrt{s_t}\sigma \cdot dZ_t + \sqrt{\varsigma_t}dZ_{j,t}\right]$$

where $Z_{j,t} \in \mathbb{R}$ is a Brownian motion specific to agent j, which is independent of $Z_{i,t}$ $(i \neq j)$ and aggregate Brownian motion $Z_t \in \mathbb{R}^d$. Idiosyncratic volatility $\varsigma_t \in \mathbb{R}$ and expected capital growth $g_t \in \mathbb{R}$ follow

$$dg_t = \lambda_g\left(\bar{g} - g_t\right)dt + \sqrt{s_t}\sigma_g \cdot dZ_t$$
$$d\varsigma_t = \lambda_s\left(\bar{\varsigma} - \varsigma_t\right)dt + \sqrt{\varsigma_t}\sigma_\varsigma \cdot dZ_t$$

Financial markets. In addition to the financial instruments in the baseline model, we consider a derivative market to trade aggregate risk. However, agents must retain at least fraction $\underline{\chi}$ of the risks (both aggregate and idiosyncratic) associated to the capital they manage. The following markets capture this set-up.

Households have access to a set of derivatives with unitary exposure to aggregate risks. The return process for the instruments with exposure to real aggregate shocks is $\left(r_t^f + \pi_t\right)dt + dZ_t$ where $\pi \in \mathbb{R}^d$ is the market price of these risks and r_t^f represents the risk-free rate as priced by households. The corresponding return process for the instrument with exposure to panics is $\left(r_t^f + \alpha_t\right)dt - dJ_t$ where $\alpha \in \mathbb{R}$ is the price of this risk. Buying the latter instrument is effectively selling insurance against self-fulfilled panics.

Households constitute the demand in the derivative market, the supply comes a securitization process. Experts securitize fraction $(1-\chi)$ of their capital holdings, respecting the financial friction: $\chi \geq \underline{\chi}$. The return process of securitized capital (i.e., the asset experts sell to financial markets) is

$$\left(r_t^f + \pi_t \cdot \left(\sigma_{k,t} + \sigma_{q,t}\right) + \alpha_t \ell_{q,t}\right)dt + \left(\sigma_{k,t} + \sigma_{q,t}\right) \cdot dZ_t - \ell_{q,t}dJ_t$$

The latter equations simply states that securitized capital is priced according market risk prices $\{\pi_t, \alpha_t\}$. Markets are able to diversify idiosyncratic risk (i.e., its market risk price is zero).

Transfer policy. After a self-fulfilled panic, failed experts receive a transfer, which is financed by taxes on the household sector. This transfer allows them to resume operations. Policy-makers set the aggregate value of the transfer $\tau_t(q_t k_t)$. Taxes and transfers are proportional to the net worth agents had before the self-fulfilled panic materializes. Let $\tau_{e,t}$ denote the transfer per unit of net worth to experts and $\tau_{h,t}$ the tax per unit of net worth levied on households. Government runs a balance budget every period. This ingredient facilitates welfare analysis, because it bounds experts losses in terms of welfare in case of a panic.

2 Solving the model

2.1 Aggregate states

Denote aggregate states as $X \equiv (w, s, \varsigma, g)$, where w corresponds to the wealth share of experts. Let $X_{-w} \equiv (s, \varsigma, g)$ refer to the exogenous aggregate states. I switch to recursive notation from now on. Denote $dX = \mu_X dt + \sigma_X dZ - \ell_X dX$ where

$$\begin{aligned}
\mu_X &= \left(w\mu_w, \lambda_g(\bar{g} - g), \lambda_s(\bar{s} - s), \lambda_\varsigma(\bar{\varsigma} - \varsigma)\right) \\
\sigma_X &= \left(w\sigma_w, \sqrt{s}\sigma_g, \sqrt{s}\sigma_s, \sqrt{\varsigma}\sigma_\varsigma\right) \\
\ell_X &= (w\ell_w, 0, 0, 0)
\end{aligned}$$

and $\{\mu_w, \sigma_w, \ell_w\}$ are endogenous objects. Consider an analogous definition for dX_{-w}.

2.2 Agents' problems and optimal decisions

The dynamic budget constraint for agent j can be written as

$$\frac{dn_j}{n_j} = (\mu_{n,j} - \hat{c}_j)\, dt + \sigma_{n,j} \cdot dZ_t + \tilde{\sigma}_{n,j} dZ_{j,t} - \ell_{n,j} dJ_t$$

where $\tilde{\sigma}_{n,j} \in \mathbb{R}$ is the exposure to idiosyncratic risk. Abusing notation, henceforth, $j \in \{h, e\}$ denotes an arbitrary households or expert. The conjecture for capital price evolution dq_t and the expressions for capital return $dR^k_{j,t}$ are the same as in the baseline model.

Households. Choose consumption $\hat{c}_h$, investment rate ι_h, capital $\hat{k}_h$, derivatives $\theta \in \mathbb{R}^d$ and $\omega \in \mathbb{R}$, and experts' debt. The dynamic budget constraint becomes

$$\begin{aligned}
\mu_{n,h} &= \hat{k}_h \mu_{R,h} + \theta \cdot (r^f + \pi) + \omega \left(r^f + \alpha \right) + \left(1 - \hat{k}_h - \theta - \omega \right) r \\
\sigma_{n,h} &= \hat{k}_h (\sigma_k + \sigma_q) + \theta \\
\tilde{\sigma}_{n,h} &= \hat{k}_h \sqrt{\varsigma} \\
\ell_{n,h} &= \hat{k}_h \ell_q + \left(1 - \hat{k}_h - \theta - \omega \right) \ell_b + \omega + \tau_h
\end{aligned}$$

The households' problem is to maximize (1) subject to the dynamic budget constraint, $n_h \geq 0$, and $\hat{k}_h \geq 0$.

Experts. Choose consumption $\hat{c}_e$, investment rate ι_e, capital $\hat{k}_e$, fraction of capital securitized $(1-\chi)$, and debt.

$$\begin{aligned}
\mu_{n,e} &= \hat{k}_e \mu_{R,e} - (1-\chi) \hat{k}_e \left(r^f + \pi \cdot (\sigma_k + \sigma_q) + \alpha \ell_q \right) - \left(\chi \hat{k}_e - 1 \right) r(\chi, \hat{k}_e) - \phi(1 - \vartheta(\hat{k}_e)) \Upsilon \\
\sigma_{n,e} &= \chi \hat{k}_e (\sigma_k + \sigma_q) \\
\tilde{\sigma}_{n,e} &= \chi \hat{k}_e \sqrt{\varsigma} \\
\ell_{n,e} &= \vartheta(\chi, \hat{k}_e)(1 - \tau_e) + \left(1 - \vartheta(\chi, \hat{k}_e) \right) \chi \hat{k}_e \ell_q
\end{aligned}$$

The experts' problem is to maximize (1) subject to the dynamic budget constraint, the financial friction $\chi \geq \underline{\chi}$, $n_e \geq 0$, and $\hat{k}_e \geq 0$. The interest rate households demand to hold experts' debt given its portfolio is $r(\chi, \hat{k}_e)$.

HJB equation. The HJB for agent j is

$$0 = \max \varphi_j(c, V_j) + \mathbb{E}\left[\frac{dV_j}{dt} \right]$$

Following the same steps as in the baseline model using

$$V_j(n; \xi_j(X)) = \frac{(\xi_j n)^{1-\gamma_j}}{1-\gamma_j}$$

as a guess for the value function delivers

$$0 = \max \underbrace{\frac{\rho_j}{1-\psi_j}\left\{\left(\frac{\hat{c}_j}{\xi_j}\right)^{1-\psi_j} - 1\right\} - \hat{c}_j}_{\text{consumption decision}} + \underbrace{\mu_{\xi_j} - \frac{\gamma_j}{2}\left(\left\|\sigma_{\xi_j}\right\|^2\right) - \frac{p}{1-\gamma_j}}_{\text{not influenced by individual decisions}} \tag{2}$$

$$+ \underbrace{\mu_{n,j} - \frac{\gamma_j}{2}\left(\|\sigma_{n,j}\|^2 + \tilde{\sigma}_{n,j}^2\right) - (\gamma_j - 1)\,\sigma_{n,j}\cdot\sigma_{\xi_j} + \frac{p}{1-\gamma_j}\left((1-\ell_{\xi_j})(1-\ell_{n,j})\right)^{1-\gamma_j}}_{\text{portfolio problem}}$$

Note that ξ_j summarize the value of investment opportunities for agent j.

Consumption and investment. These decisions are symmetric for both type of agents: $\hat{c}_j = \rho_j^{1/\psi_j}\xi_j^{1-1/\psi_j}$ and $\Phi'(\iota_j) = 1/q$.

Households' portfolio. Define the (aggregate) risk-adjusted capital return Δ_h as

$$\mu_{R,h} \equiv r^f + \pi\cdot(\sigma_k + \sigma_q) + \alpha\ell_q + \Delta_h$$

Then, FOCs are

$$[\theta]: \quad \pi = \gamma_h\sigma_{n,h} + (\gamma - 1)\,\sigma_X\cdot\partial_X Ln\xi_h \equiv \pi_h \tag{3a}$$

$$[\omega]: \quad \alpha = p\left(1-\ell_{n,h}\right)^{-\gamma_h}\left(1-\ell_{\xi_h}\right)^{1-\gamma_h} \equiv \alpha_h \tag{3b}$$

$$[\hat{k}_h]: \ \Delta_h \leq \hat{k}_h\gamma_h\varsigma \tag{3c}$$

where the latter holds as an equality of $\hat{k}_h > 0$. The first two conditions imply that market price of risks $\{\pi, \alpha\}$ correspond to households' valuation of them $\{\pi_h, \alpha_h\}$.

Pricing of experts' debt. The interest rate households demand to hold the debt of an expert with portfolio $(\chi, \hat{k}_e)$ respects

$$r(\chi, \hat{k}_e) = r^f + \alpha\ell_b(\chi, \hat{k}_e)$$

where

$$\ell_b(\chi, \hat{k}_e) = \left[1 - (1-\ell_q)\left(\frac{\chi\hat{k}_e}{\chi\hat{k}_e - 1}\right)\right]^+$$

are the losses for creditors in case of a self-fulfill panic. Denote the share of wealth experts invest on direct holdings of capital as $\beta_e \equiv \chi \hat{k}_e$. This excludes the capital he manages but securitizes. The recovery rate in case of bankruptcy depends only in this fraction, i.e., $\ell_b(\chi, \hat{k}_e) \equiv \ell_b(\beta_e)$, so $r(\chi, \hat{k}_e) \equiv r(\beta_e)$ and $\vartheta(\chi, \hat{k}_e) = \vartheta(\beta_e)$.

Experts' portfolio. Define the (aggregate) risk-adjusted capital return Δ_e as

$$\mu_{R,e} \equiv r^f + \pi \cdot (\sigma_k + \sigma_q) + \alpha \ell_b + \chi \Delta_e$$

Then, FOCs are

$$[\hat{k}_e]: \quad \Delta_e = (\pi_e - \pi) \cdot (\sigma_k + \sigma_q) + \chi \hat{k}_e \gamma_e \varsigma + \underbrace{\left(1 - \vartheta(\chi, \hat{k}_e)\right) \left[(\alpha_e - \alpha)\, \ell_q + \alpha \ell_d\right]}_{=0 \text{ in symmetric equilibriums}} \tag{4a}$$

$$[\chi]: \quad 0 \leq \Delta_e \tag{4b}$$

where the latter holds as an equality if $\chi > \underline{\chi}$ and

$$\alpha_e \equiv p\, (1 - \ell_{n,e})^{-\gamma_e} \left(1 - \ell_{\xi_e}\right)^{1-\gamma_e}$$
$$\pi_e \equiv \gamma_e \sigma_{n,e} + (\gamma_e - 1) \sigma'_X \partial_X Ln \xi_e$$

are the valuations of risk for experts.

2.3 Equilibrium

Market clearing. Capital

$$\hat{k}_e w + \hat{k}_h (1 - w) = 1 \tag{5}$$

for convenience denote the share of capital manage by households $\kappa \equiv \hat{k}_e w$.

Goods

$$q \left[(1-w)\, \hat{c}_h + w \hat{c}_e\right] + \iota(q) = a_e \kappa + a_h (1 - \kappa) \tag{6}$$

Derivatives

$$(1 - \chi) \kappa \,(\sigma_k + \sigma_q) = (1 - w)\, \theta \tag{7a}$$

$$(1 - \chi) \kappa \ell_q = (1 - w)\, \omega \tag{7b}$$

where the LHS is the supply from securitized capital, and the RHS corresponds to the demand from households. These conditions are equivalent to

$$(1-w)\ell_{n,h} + w\ell_{n,e} = \ell_q$$
$$(1-w)\sigma_{n,h} + w\sigma_{n,e} = \sigma_k + \sigma_q$$

where the RHS is the total risk in the economy, and the LHS corresponds to the split of these risks between the two type of agents.

Government balance budget

$$\tau = (1-w)\tau_h = w\tau_e$$

Law of motion of aggregate states. Replacing optimal decision into the dynamic budget constraints delivers

$$\mu_{n,h} = r^f + \pi \cdot \sigma_{n,h} + \alpha\left(\ell_{n,h} - \tau_h\right) + \hat{k}_h \Delta_h \tag{8}$$
$$\mu_{n,e} = r^f + \pi \cdot \sigma_{n,e} + \alpha\left(\ell_{n,e} + \tau_e \vartheta\right) + \beta_e \Delta_e + \underbrace{\alpha\vartheta\left(\beta_e \ell_q - 1\right) - \alpha\ell_b\left(\beta_e - 1\right)}_{=0 \text{ in symmetric equilibriums}} \tag{9}$$

where the latter term disappears in symmetric equilibriums: either experts choose a safe portfolio $\vartheta(\beta_e) = 0$ and there is no exposure to panics $p = \alpha = 0$, or experts choose a risky portfolio $\vartheta(\beta_e) = 1$ and the compensation for losses beyond limited liability is offset by the extra interest rate payments due to the potential bankruptcy. From now on, I only consider symmetric equilibria among experts.

Replacing the dynamic budget constraints and market clearing conditions into the law of motion for experts' wealth share (derived from Ito's formula for $w \equiv N_e/(N_e + N_h)$) delivers

$$w\mu_w = w(1-w)\left(\beta_e \Delta^e - \hat{k}_h \Delta^h - \hat{c}_e + \hat{c}_h\right) + \alpha\left(\ell_q - \ell_b\right)\left(\beta_e - w\right) + \left(\pi - (\sigma_k + \sigma_q)\right) w\sigma_w + \lambda_d(\nu - w)$$
$$w\sigma_w = (\chi\kappa - w)\left(\sigma_k + \sigma_q\right)$$
$$w\ell_w = w - \tilde{w}$$

where $\tilde{w}$ solves

$$\tilde{w} = \tau(X)$$

Consistency conditions. Capital price and investment opportunities ξ_j for $j = \{h, e\}$ must be consistent with the evolution of aggregate states. These conditions follow directly from applying Ito's lemma to functions $q(X)$ and $\xi_j(X)$:

$$\mu_q = \frac{1}{q}\left[\mu_X \cdot \partial_X q + \frac{1}{2}\text{tr}\left(\sigma_X' \partial_{XX'} q \sigma_X\right)\right] \tag{11a}$$

$$\sigma_q = \frac{(\chi\kappa - w)\sigma_K \partial_w \ln q + \sigma_{X_{-w}}' \partial_{X_{-w}} \ln q}{1 - (\chi\kappa - w)\partial_w \ln q} \tag{11b}$$

$$\ell_q = 1 - q(\tilde{w}, X_{-w})/q \tag{11c}$$

and

$$\mu_{\xi,j} = \mu_X \partial \ln \xi_j + \frac{1}{2} tr\left[\sigma_X' \left(\partial_{XX'} \ln \xi_j + \partial_X \ln \xi_j \partial_X \ln \xi_j'\right)\sigma_X\right] \tag{12a}$$

$$\sigma_{\xi,j} = \sigma_X' \partial \ln \xi_j \tag{12b}$$

$$\ell_{\xi,j} = 1 - \xi_j(\tilde{w}, X_{-w})/\xi_j \tag{12c}$$

HJB equations. Replace consumption and investment decision into HJB (2) delivers

$$\begin{aligned} 0 = &\frac{1}{1-\psi_j}\rho_j^{1/\psi_j}\xi_j^{1-1/\psi_j} - \frac{\rho_j}{1-\psi_j} + \mu_{n,j} - (\gamma_j - 1)\,\sigma_{n,j}\cdot\sigma_{\xi_j} - \frac{\gamma_j}{2}\left(\|\sigma_{n,j}\|^2 + \tilde{\sigma}_{n,j}^2\right) \\ &+ \frac{p}{1-\gamma_j}\left[\left(\left(1-\ell_{\xi_j}\right)(1-\ell_{n,j})\right)^{1-\gamma_j} - 1\right] + \mu_{\xi_j} - \frac{\gamma_j}{2}\left\|\sigma_{\xi_j}\right\|^2 \end{aligned} \tag{13}$$

where $\mu_{n,j}$ is given by (8) for households or (9) for experts, and

$$\sigma_{n,h} = \frac{1-\chi\kappa}{1-w}(\sigma_k + \sigma_q), \quad \tilde{\sigma}_{n,h} = \frac{1-\kappa}{1-w}\sqrt{\varsigma}$$

$$\sigma_{n,e} = \frac{\chi\kappa}{w}(\sigma_k + \sigma_q), \quad \tilde{\sigma}_{n,e} = \frac{\chi\kappa}{w}\sqrt{\varsigma}$$

$$\ell_{n,h} = \tau_h + \left(\frac{1-\chi\kappa}{1-w}\right)\ell_q + \left(\frac{\chi\kappa - w}{1-w}\right)\ell_b$$

$$\ell_{n,e} = \Upsilon(1-\tau_e) + (1-\Upsilon)\beta_e\ell_q$$

which follow directly from market clearing conditions.

Definition. *A Markov equilibrium in* $X \equiv (w, s, \varsigma, g)$ *is a set of functions: aggregate outcomes and their dynamics* $r^f, \pi, \alpha, q, \mu_q, \sigma_q, \ell_q, \ell_b, p$*, portfolio decisions* $\hat{k}_e, \chi, \hat{k}_h, \theta, \omega$*, investment opportunities* $\xi_j, \mu_{\xi,j}, \sigma_{\xi,j}, \ell_{\xi,j}$ *for* $j \in \{e, h\}$*, and aggregate states' dynamics* μ_X, σ_X, ℓ_X *such that:*

1. *Given aggregate outcomes and the evolution of aggregate states,* $\hat{k}_h, \theta, \omega$ *and* $\hat{k}_e, \chi$ *solve portfolio problems for households and experts, respectively: (3) and (4). Investment opportunities* $\xi_j, \mu_{\xi,j}, \sigma_{\xi,j}, \ell_{\xi,j}$ *satisfy the HJB equation (13) for* $j = \{h, e\}$*.*
2. *Markets clear: (5), (6), (7).*
3. *The law of motion of* w *satisfies (10).*
4. *Consistency conditions hold: (12) for* $j = \{h, e\}$*, (11) and* $p = \Gamma 1_{\{\ell_b > 0\}}$*.*

3 Invariance results

Theorem 1' *(Relative capital demand's invariance to panics)*. *The relative capital demand of experts over households is independent of the vulnerability to panics (i.e., taken as given aggregate outcomes,* $\hat{k}_e - \hat{k}_h$ *is the same as in the model without panics* $p \equiv 0$*).*

Proof. Subtract the FOC for capital of experts (4a) and households (3c), then relative capital demand

$$\frac{a_e - a_h}{q} \geq \chi(\pi_e - \pi) \cdot (\sigma_k + \sigma_q) + \left(\chi \gamma_e \tilde{\sigma}_{n,e} - \gamma_h \tilde{\sigma}_{n,h}\right)\sqrt{\varsigma}$$

which holds with equality if $\kappa < 1$. Then, conditional on aggregate outcomes, the relative capital demand is independent of p.

Theorem 1 *(Capital's invariance to panics).* *Consider the generalized model with logarithmic preferences (i.e.,* $\gamma_j = \psi_j = 1$ *for* $j \in \{h, e\}$*). Then, capital allocation* κ *is invariant to financial panics, i.e., equilibrium function* $\kappa(w, s)$ *is the same as in the model without panics (*$p \equiv 0$*). The same applies to capital price* $q(w, s)$ *and capital allocation efficiency* $a(w, s)$*.*

Proof. With logarithmic preferences, the characterization of a Markov equilibrium can be divided in two blocks, where one of them is independent of investment opportunities $\{\xi_h, \xi_e\}$.

This block correspond to the following equations:

$$\Delta_e = \Big(\sigma_{n,e} - \sigma_{n,h}\Big) \cdot (\sigma_k + \sigma_q) + \beta_e \varsigma \tag{14a}$$

$$0 \leq \Delta_e \quad \text{with equality if } \underline{\chi} < \chi \tag{14b}$$

$$\Delta_h \leq \hat{k}_h \varsigma \quad \text{with equality if } 0 < \hat{k}_h \tag{14c}$$

$$\frac{a_e - a_h}{q} \geq \chi \Delta_e - \Delta_h \quad \text{with equality if } 0 < \hat{k}_h \tag{14d}$$

$$\rho q + \iota(q) = (a_e - a_h)\kappa + a_h \tag{14e}$$

$$\sigma_k + \sigma_q = \frac{\sigma_k + \sigma'_{X_{-w}} \partial_{X_{-w}} \ln q}{1 - (\chi\kappa - w)\partial_w \ln q} \tag{14f}$$

This system implicitly defines the PDE for capital $q(X)$: we can solve statically for $\{\Delta_e, \Delta_h, \kappa, \chi, \sigma_q\}$ as a function of q and its (first-order) derivatives and the boundary conditions are $q(0, X_{-w}) = q^{\dagger}$ and $q(1, X_{-w}) = q^{\star}$ solved from (14e) with $\kappa = 0$ and $\kappa = 1$, respectively. Hence, the system fully characterizes function $q(X)$ and it does not vary if we consider the case without run, i.e., $\Gamma \equiv 0$.