Dynamic Self-Fulfilling Fire Sales*

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February 13, 2025

Abstract

Why do fire sales occur if many risks are hedgeable? We study a version of Brunnermeier and Sannikov (2014) in which all fundamental risks can be hedged frictionlessly. Our analysis shows that fire sales are inherently self-fulfilling. Fundamental shocks can never cause fire sales, and an efficient, safe equilibrium exists. On the other hand, there exists an equilibrium in which agents coordinate fire sales on non-fundamental shocks. A simple refinement based on vanishingly-small perceived fundamental risk eliminates the safe equilibrium and selects the fire sale equilibrium as the unique outcome.

JEL Codes: E00, E44, G01.

Keywords: financial frictions, fire sales, self-fulfilling equilibria, financial crises, hedging.

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Why do fire sales occur if many risks are hedgeable? Why is risk concentrated on the balance sheets of intermediaries and productive experts? Which types of risks are concentrated? Our paper provides a new, self-fulfilling perspective on these questions.

We study a canonical macro-finance model, similar to Kiyotaki and Moore (1997) and Brunnermeier and Sannikov (2014), that has launched a large subsequent literature. The core mechanism in such models is sometimes referred to as a "financial accelerator" because the economic impact of fundamental shocks is amplified by imperfect risk-sharing.[1] Our departure from these models: we allow agents to frictionlessly hedge all fundamental risks. This feature brings our analysis closer to Di Tella (2017).

As a general principle, when a risk is shared, the accelerator mechanism breaks down. To re-open the door to accelerator-type fluctuations, we limit risk-sharing in a novel way. While our model features markets to hedge all *fundamental* shocks, markets do not exist for hedging certain endogenous risks that emerge in a self-fulfilling manner, which we will refer to as "sunspot shocks." Sunspot shocks are non-fundamental, but agents could potentially coordinate on them. Such coordination affects equilibrium precisely because sunspot shocks are not directly hedgeable.

What emerges are self-fulfilling fire sales. More productive agents ("experts") manage a disproportionate share of productive capital and share its fundamental risks perfectly with everyone else ("households"). In this world, fundamental shocks cannot cause amplification. But equilibrium is not necessarily efficient or smooth. If a sunspot shock arrives, experts may coordinate to sell capital to households. If such coordinated selling occurs, the capital price would fall in equilibrium, due to the lower productive efficiency of households. From the perspective of an individual expert, the selling by other experts impinges her wealth and motivates her to also sell capital. The dynamics at play are very similar to runs, but stem from the asset side rather than liability side.

Our main result proves and constructs such an equilibrium. In fact, we show that there are a continuum of "partial fire sale" equilibria as well, depending on how agents coordinate on the tail-event price of capital. Our benchmark sunspot equilibrium is a "full fire sale" equilibrium, in which the tail-event capital price is its worst-case value.

We then go on to demonstrate how our benchmark sunspot equilibrium is a limiting

[1] Extensions of this framework have been used to study idiosyncratic uncertainty shocks (Di Tella, 2017); shadow banking (Moreira and Savov, 2017); bank capital regulation (Phelan, 2016; Klimenko et al., 2017); monetary policy and liquidity (Drechsler et al., 2018; d'Avernas and Vandeweyer, 2023); quantitative easing (Silva, 2024); optimal policy (Di Tella, 2019); the quantitative frequency and severity of crises (He and Krishnamurthy, 2019); bank runs (Gertler and Kiyotaki, 2015; Gertler et al., 2020; Mendo, 2020); extrapolative sentiments (Krishnamurthy and Li, 2024; Maxted, 2024); time-varying diversification (Khorrami, 2021); and long-run risks (Hansen et al., 2024). On the asset-pricing side, this literature is often referred to as "intermediary asset pricing" (He and Krishnamurthy, 2012, 2013).

case of the conventional equilibrium. Imagine fundamental risks were not hedgeable, and denote their volatility σ. As $\sigma \to 0$, the conventional financial accelerator equilibrium converges to an equilibrium with volatility (formalizing the "volatility paradox" by Brunnermeier and Sannikov, 2014). Agents continue to coordinate on the fundamental shock, even though it has zero volatility, and the result is identical to our benchmark sunspot equilibrium.

Finally, we provide a very simple refinement that ensures uniqueness. Our baseline model features an indeterminate degree of fire sales, including the possibility of an efficient equilibrium with no fire sales. Why would agents coordinate on a particular fire-sale equilibrium? Imagine the sunspot shock, which is not hedgeable, is perceived by agents to have a vanishingly-small fundamental impact ς. Imagine furthermore that agents face a leverage cap of β. In this environment, loosening the leverage cap ($\beta \to \infty$) and then eliminating the perceived risk ($\varsigma \to 0$) selects our benchmark fire-sale equilibrium as the unique outcome.

These results overturn a conventional wisdom that the financial accelerator mechanism breaks down when fundamental shocks are hedgeable. So long as agents cannot hedge *every conceivable shock*, the door remains open to accelerator-type fluctuations and, in fact, such fluctuations are selected as the unique outcome in our context. Our analysis also clarifies that fire-sale dynamics are likely to be driven by non-fundamental shocks, especially emergent shocks that lack developed hedging markets.

Related literature. In addition to the financial accelerator papers cited above, our paper contributes to two literatures: (i) the literature on sunspot fluctuations and (ii) the literature on the accelerator mechanism in the presence of hedging markets.

Beginning with Azariadis (1981) and Cass and Shell (1983), the literature on sunspot fluctuations often appeals to overlapping generations as a market access friction. At a very high level, this bears similarity to our model, because some type of incomplete markets is critical to multiplicity. Like us, these papers, and their more recent articulations (Farmer, 2018; Gârleanu and Panageas, 2021), often feature wealth redistribution as a key mechanism. A key difference between our model and these OLG models is that, for most specifications, they feature a multiplicity of fundamental or "certainty" equilibria, and the sunspot equilibria are built as lotteries over the fundamental equilibria; this type of construction characterizes the vast majority of non-OLG sunspot papers as well.[2] By contrast, our model has a unique fundamental equilibrium. Also distinct is our

[2]One famous exception is the appendix of the aforementioned paper by Cass and Shell (1983). In their construction, risk-preference heterogeneity among agents interacts with the sunspot risk to create a self-fulfilling redistribution. Our paper differs from Cass and Shell's appendix in several dimensions. First, we

refinement argument that eliminates all equilibria but the worst sunspot equilibrium.

Bank runs, financial panics, and sudden stops represent a particular type of sunspot fluctuation. While closely related, our dynamics are distinct from runs: whereas bank runs and its cousins are liability-side phenomena, self-fulfilled fire sales are asset-side phenomena. Furthermore, our fire sales do not require asset-market illiquidity or maturity mismatch (Diamond and Dybvig, 1983). Thus, our paper shows how run-like dynamics can occur in a broader set of environments.

Our companion paper Khorrami and Mendo (2024) studies a similar framework and provides the complementary analysis of sunspot equilibria that are not Markovian in the wealth distribution. In particular, whereas the present paper treats the asset price q as a function of experts' wealth share η, our other paper allows q to be driven by additional non-fundamental state variables ("sentiments"). This leads to a dramatically different analysis, in which "stochastic stability" becomes the key criterion for whether or not a sunspot equilibrium can exist. Here, imposing the conventional Markovian assumption $q = q(\eta)$ allows us to say much more about the sunspot equilibrium, including its uniqueness and similarity to the conventional accelerator equilibrium. In short, our two papers study distinct classes of sunspot equilibria. (Another substantive difference: Khorrami and Mendo, 2024, does not permit full hedging of fundamental shocks.)

Finally, the current paper contributes to the literature on the impact of hedging on the financial accelerator. As is well understood, the conventional accelerator mechanism typically breaks down under financial markets for aggregate risk-sharing (Krishnamurthy, 2003; Di Tella, 2017). If all aggregate risks can be shared, then accelerator-like fluctuations can only emerge under a combination of some type of financial friction and induced heterogeneous risk preferences. For example, Di Tella (2017) shows that the combination of equity-issuance constraints (due to moral hazard) and Epstein-Zin preferences can induce experts to hold concentrated exposure to idiosyncratic volatility shocks. Bocola and Lorenzoni (2023) show that the combination of borrowing constraints (due to limited enforcement) and non-expert hedging demand (due to their risk aversion and claim to risky labor income) can induce experts to hold concentrated exposure to any aggregate shock. Our paper lies somewhere in between these latter papers and the conventional accelerator literature: we assume fundamental shocks are perfectly hedgeable

feature productive heterogeneity rather than preference heterogeneity, so our sunspot fluctuations have real effects. Because of these real effects, our self-fulfilling sunspot equilibria are inefficient, whereas they are "dynamically Pareto efficient" in Cass and Shell (1983). Second, our model is necessarily fully dynamic: any finite-horizon version of our model could not support multiplicity, by backward-induction. There are also minor differences such as the fact that such an equilibrium in Cass and Shell (1983) requires at least 3 agent types and at least 2 consumption goods. See also Bacchetta et al. (2012) and Benhabib et al. (2015) for examples of sunspot equilibria arising despite a unique fundamental equilibrium.

(and so they cannot cause any amplification), while sunspot shocks are not hedgeable. This feature relates to Dávila and Philippon (2017), who model "incompleteness shocks" as a way to analyze an intermediate level of market completeness.

1 Model

The model structure is the same as in Khorrami and Mendo (2024), which is a simplified version of Brunnermeier and Sannikov (2014) that does not include capital investment.

Information structure. There are two types of uncertainty in the economy, modeled as independent Brownian motions (W, Z). The *fundamental shock* W directly impacts capital, whereas the second shock Z is a *sunspot shock* that is extrinsic to economic primitives. Section 4 studies Poisson jump shocks, rather than Brownian shocks.

Technology and markets. There are two goods, non-durable consumption and durable capital that produces consumption. When an individual agent i holds capital $k_{i,t}$, it grows exogenously as

$$dk_{i,t} = k_{i,t}[gdt + \sigma dW_t], \tag{1}$$

where g, $\sigma > 0$ are exogenous constants. The capital-quality shock σdW introduces fundamental randomness in technology. The relative price of capital is denoted by q_t and is determined in equilibrium. (Note that (1) excludes the effect of capital trades.)

There are two agent types, experts and households, who differ in their productivity. Experts produce a_e units of output per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$. Because all agents of the same type will ultimately behave as scaled versions of each other, we index agents $i \in \{e, h\}$ simply by their type.

Financial markets consist of a short-term, risk-free bond in zero net supply that pays interest rate r_t and a financial market for contracting on the fundamental shock, which offers expected return $\pi_t dt$ per unit of exposure to dW_t. The financial friction is that agents cannot issue equity nor state-contingent debt when managing capital.[3]

Preferences and optimization. Given the stated assumptions, we can write the dynamic

[3] Partial equity issuance, with some constraint, will generate similar results. See Online Appendix F.

budget constraint of any agent of type $i \in \{e, h\}$ as

$$dn_{i,t} = \underbrace{\Big[(n_{i,t} - q_t k_{i,t}) r_t - c_{i,t}\Big] dt}_{\text{consumption-savings}} + \underbrace{q_t k_{i,t} \Big[\frac{a_i}{q_t} dt + \frac{d(q_t k_{i,t})}{q_t k_{i,t}}\Big]}_{\text{capital returns}} + \underbrace{x_{i,t} \big[\pi_t dt + dW_t\big]}_{\text{financial hedges}}, \tag{2}$$

where n is the agent's net worth, c is consumption, k is capital holdings, and x denotes hedging positions. Brunnermeier and Sannikov (2014) effectively imposes $x \equiv 0$ as a constraint (as does our companion paper Khorrami and Mendo, 2024).

Experts and households have logarithmic utility, with discount rates ρ_e and $\rho_h < \rho_e$, respectively. In the online appendix, we generalize to CRRA utility with alternative levels of risk aversion. Experts' higher discount rate ensures a stationary wealth distribution. Agents solve

$$\sup_{\substack{(c_{i,t}, k_{i,t}, n_{i,t}, x_{i,t})_{t \geq 0} \\ c_{i,t} \geq 0,\, k_{i,t} \geq 0,\, n_{i,t} \geq 0}} \mathbb{E}\Big[\int_0^\infty e^{-\rho_i t} \log(c_{i,t}) dt\Big] \tag{3}$$

subject to (2) and given $n_{i,0}$. The solvency constraint $n_{i,t} \geq 0$ is the natural borrowing limit, given the absence of labor income. Problem (3) is homogeneous in (c, k, n, x), so we think of the expert and household as representative agents in their class.

Definition 1. For initial endowments $k_{e,0}, k_{h,0} > 0$ such that $k_{e,0} + k_{h,0} = K_0$, an *equilibrium* consists of stochastic processes—adapted to the filtered probability space generated by $\{W_t, Z_t : t \geq 0\}$—for capital price q_t, interest rate r_t, risk price π_t, capital holdings $(k_{e,t}, k_{h,t})$, hedges $(x_{e,t}, x_{h,t})$, consumptions $(c_{e,t}, c_{h,t})$, and net worths $(n_{e,t}, n_{h,t})$, such that:

(i) initial net worths satisfy $n_{e,0} = q_0 k_{e,0}$ and $n_{h,0} = q_0 k_{h,0}$;

(ii) taking processes (q, r, π) as given, agents solve (3) subject to (2);

(iii) consumption, capital, and hedging markets clear at all dates, i.e.,

$$c_{e,t} + c_{h,t} = a_e k_{e,t} + a_h k_{h,t} \tag{4}$$

$$k_{e,t} + k_{h,t} = K_t \tag{5}$$

$$x_{e,t} + x_{h,t} = 0, \tag{6}$$

where K_t follows the same dynamics as those given in (1).

Note that each agent is endowed with positive initial capital ($k_{i,0} > 0$), so that they have positive initial wealth ($n_{i,0} > 0$) and can thus consume over their lifetime.

1.1 Equilibrium characterization

We start with a useful equilibrium characterization. First, conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dW_t + \varsigma_{q,t}dZ_t]. \tag{7}$$

There are two potential avenues for random fluctuations. The standard term σ_q represents amplification (or dampening) of fundamental shocks, as in Brunnermeier and Sannikov (2014) and others. By contrast, ς_q measures sunspot volatility that only exists because agents believe in it.

Given log utility and the scale-invariance of agents' budget sets, individual optimization problems are readily solvable. Optimal consumption satisfies the standard formula $c_{i,t} = \rho_i n_{i,t}$. Capital holdings and financial hedges are determined via a mean-variance problem (see Appendix A):

$$\max_{k\geq 0, x\in\mathbb{R}} \Big\{ \mathbb{E}[\frac{dn}{n}] - \frac{1}{2}\mathrm{Var}[\frac{dn}{n}] \Big\} \tag{8}$$

Plugging in capital and price dynamics in the dynamic wealth equation (2), and rearranging, this problem becomes

$$\max_{\tilde{k}\geq 0, \tilde{x}\in\mathbb{R}} \Big\{ \tilde{k}\Big(\frac{a}{q} + g + \mu_q + \sigma\sigma_q - (\sigma+\sigma_q)\pi - r\Big) + \tilde{x}\pi - \frac{1}{2}\big(\tilde{k}\varsigma_q\big)^2 - \frac{1}{2}\tilde{x}^2 \Big\},$$

where $\tilde{k} := \frac{qk}{n}$ and $\tilde{x} := \frac{x}{n} + \frac{qk}{n}(\sigma + \sigma_q)$ are the agent's per-unit-of-wealth exposures to the sunspot shock $\varsigma_q dZ$ and fundamental shock dW, respectively. Note that $\tilde{x}$ is unconstrained because x is unconstrained. The optimality conditions are

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q - (\sigma+\sigma_q)\pi - r = \frac{qk_e}{n_e}\varsigma_q^2 \tag{9}$$

$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q - (\sigma+\sigma_q)\pi - r \leq \frac{qk_h}{n_h}\varsigma_q^2 \quad \text{(with equality if } k_h > 0\text{)} \tag{10}$$

for capital holdings and

$$\pi - \frac{qk_e}{n_e}(\sigma+\sigma_q) = \frac{x_e}{n_e} \tag{11}$$

$$\pi - \frac{qk_h}{n_h}(\sigma+\sigma_q) = \frac{x_h}{n_h} \tag{12}$$

for hedges. (Note that experts' capital optimality condition (9) assumes the solution is interior, i.e., $k_e > 0$. But this is clearly required in any equilibrium given experts earn a strictly higher expected return than households.) These conditions fully summarize optimality.[4]

Next, we aggregate. Due to financial frictions and productivity heterogeneity, both the distribution of wealth and capital holdings will matter in equilibrium. Define experts' wealth and capital shares:

$$\eta := \frac{n_e}{n_e + n_h} = \frac{n_e}{qK} \quad \text{and} \quad \kappa := \frac{k_e}{K}.$$

Given agents' solvency and capital short-sales constraints, we must have $\eta \in [0,1]$ and $\kappa \in [0,1]$ in equilibrium. Substitute optimal consumption into goods market clearing (4), divide by aggregate capital K, and use the definitions of η and κ, to obtain

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h, \tag{PO}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1-\eta)\rho_h$ is the wealth-weighted average discount rate. Equation (PO) connects asset price q to output efficiency κ, which we call a *price-output* relation.

Using the definitions of η and κ, experts' and households' portfolio shares can be written $\frac{qk_e}{n_e} = \frac{\kappa}{\eta}$ and $\frac{qk_h}{n_h} = \frac{1-\kappa}{1-\eta}$. Then, differencing the optimal portfolio conditions (9)-(10), we obtain the *risk-balance* condition

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\varsigma_q^2\Big]. \tag{RB}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the differential return of experts over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $\frac{\kappa-\eta}{\eta(1-\eta)}\varsigma_q^2$. Summing portfolio conditions (9)-(10), weighted by κ and $1-\kappa$, yields an equation for r:

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q - (\sigma + \sigma_q)\pi - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)\varsigma_q^2. \tag{13}$$

Combining optimal financial hedges (11)-(12) with the zero net supply condition (6), we

[4] The only additional optimality conditions are the transversality conditions $\lim_{T\to\infty} \mathbb{E}[e^{-\rho_i T}\frac{1}{c_{i,T}}n_{i,T}] = 0$. However, using $c_i = \rho_i n_i$, we see that transversality automatically holds.

obtain a full risk-sharing result for the equilibrium risk price

$$\pi = \sigma + \sigma_q. \tag{14}$$

Finally, applying Itô's formula to η, and using net worth dynamics (2), wealth share dynamics are given by

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}dW_t + \varsigma_{\eta,t}dZ_t, \quad \text{given} \quad \eta_0, \tag{15}$$

where [5]

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}\varsigma_q^2 \tag{16}$$

$$\sigma_\eta = 0 \tag{17}$$

$$\varsigma_\eta = (\kappa - \eta)\varsigma_q. \tag{18}$$

Note the full risk-sharing result $\sigma_\eta = 0$ on fundamental risk. Also, the initial wealth distribution $\eta_0 = \frac{n_{e,0}}{q_0 K_0} = \frac{k_{e,0}}{K_0}$ is pinned down uniquely by capital endowments. Moreover, $\eta_0 \in (0,1)$ because both experts and households start with positive initial capital.

1.2 Fundamental Equilibrium

We focus on equilibria that are Markov in the state variable η (Khorrami and Mendo 2024 analyze equilibria with additional "sentiment states" beyond η). Going forward, "equilibrium" refers to this Markovian subclass. Among those, categorize equilibria into fundamental or sunspot, depending on whether dZ matters.

[5] To derive this, start from the definition $\eta := n_e/(n_e + n_h)$. Applying Itô's formula yields

$$d\eta = \eta(1-\eta)\Big(\frac{dn_e}{n_e} - \frac{dn_h}{n_h}\Big) - \eta(1-\eta)\Big(\eta\frac{d[n_e]}{n_e^2} - (1-\eta)\frac{d[n_h]}{n_h^2} + (1-2\eta)\frac{d[n_e, n_h]}{n_e n_h}\Big),$$

where $d[x]$ and $d[x,y]$ denote the quadratic variation of x and quadratic co-variation of x with y, respectively. Finally, plug in agents consumption and portfolio choices into their net worth evolution (2), and use the aggregate variables η and κ, to obtain

$$\frac{dn_e}{n_e} = \Big[r - \rho_e + \frac{\kappa}{\eta}(\mu_R^e - r - (\sigma + \sigma_q)\pi) + \pi^2\Big]dt + \pi dW + \frac{\kappa}{\eta}\varsigma_q dZ$$
$$\frac{dn_h}{n_h} = \Big[r - \rho_h + \frac{1-\kappa}{1-\eta}(\mu_R^h - r - (\sigma + \sigma_q)\pi) + \pi^2\Big]dt + \pi dW + \frac{1-\kappa}{1-\eta}\varsigma_q dZ$$

where $\mu_R^e := \frac{a_e}{q} + g + \mu_q + \sigma\sigma_q$ and $\mu_R^h := \frac{a_h}{q} + g + \mu_q + \sigma\sigma_q$ denote experts' and households' expected return from managing capital, respectively. Combining these results yields (16)-(18).

Definition 2. A *Fundamental Equilibrium* is an equilibrium in which $\varsigma_q \equiv 0$. Any other equilibrium is called a *Brownian Sunspot Equilibrium* (BSE).

The Fundamental Equilibrium is unique, efficient, and deterministic. Allocative efficiency is immediately evident: if $\varsigma_q \equiv 0$, then (RB) implies $\kappa = 1$ forever. Risk-sharing is also efficient, captured by deterministic relative wealth dynamics: from (17)-(18), and using $\varsigma_q = 0$, we have that $\sigma_\eta = 0$ and $\varsigma_\eta = (\kappa - \eta)\varsigma_q = 0$. In fact, the equilibrium obeys the following deterministic dynamics:

$$\frac{\dot{q}_t}{q_t} = \eta_t(1-\eta_t)\frac{(\rho_e - \rho_h)^2}{\eta_t\rho_e + (1-\eta_t)\rho_h} \tag{19}$$

$$\dot{\eta}_t = -\eta_t(1-\eta_t)(\rho_e - \rho_h) \tag{20}$$

(To obtain these equations, substitute $\kappa = 1$ into equation (PO) to get $q_t = a_e/\bar{\rho}(\eta_t)$, and combine with the previous results.) Finally, because η_0 is given and $q = a_e/\bar{\rho}(\eta)$ is purely a function of η, this Fundamental Equilibrium is *unique*: its initial condition is pinned down, and its dynamics are given uniquely by (19)-(20). Therefore, none of our results arise due to a multiplicity without sunspot shocks, unlike many classical sunspot equilibrium constructions.

Lemma 1 (Fundamental Equilibrium)**.** *There exists a unique Fundamental Equilibrium in which experts manage all capital, $\kappa = 1$, and its price $q_t = a_e/\bar{\rho}(\eta_t)$ evolves deterministically.*

1.3 Discussion of setup and assumptions

Before proceeding to our main results, we pause to discuss some modeling choices and assumptions.

First, why do we include both fundamental and sunspot shocks? For instance, one possible alternative setup is to feature a single shock, with fundamental volatility σ, and to interpret different values of σ as different financial market structures. The conventional financial accelerator model is captured by $\sigma > 0$. Conversely, $\sigma = 0$ means the single shock is itself a sunspot shock, and the reader could interpret this environment as analogous to our present model where all fundamental sources of uncertainty are hedgeable. In fact, this analogy is apt: given perfect hedging markets for the fundamental shock dW, it cannot play any role in fluctuations, as $\sigma_\eta = 0$ in equation (17). Therefore, one may view the inclusion of two shocks as a distraction.

We choose to adopt our two-shock model for both pedagogical reasons and for streamlined interpretation. Pedagogically, we think there is some value in developing the argument for why the fundamental shock cannot matter but the sunspot shock can.[6] Including two shocks also maintains expositional clarity. A key motivation for this paper is the idea that emergent shocks, for which hedging markets are not yet developed, can impact the economy. It is thus convenient to have a shock, the sunspot shock dZ, that always represents this emergent but extraneous source of uncertainty, regardless of the parameters. This keeps our model closer, in a literal sense, to the language we use throughout the introduction.

As we will show below, the sunspot shock dZ can affect the equilibrium because there are no markets for hedging its impact. How should the reader interpret this assumption? In our view, there are two essential characteristics that such a shock should possess: (1) it should be relatively novel, and (2) it should affect aspects of the economy which are not as liquid in asset markets. First, if a shock is old and relatively well-understood, e.g., interest rate risk, then direct futures markets can be used to hedge such risks. By contrast, it may have been difficult to directly hedge the proliferation of mortgage derivatives in the run up to the 2008 financial crisis, because those were relatively new assets. Second, if a shock only affects publicly-traded firms like the big banks, then an easy way to hedge it would be to short the market index of these big banks. In our model, the ability to short the "market index" of other experts' equity would suffice to complete markets even if individual experts' equity-issuance was limited. By contrast, if some experts' equity is non-tradeable as in the case of private firms, then trading the public experts' equity would only serve as an approximate, but imperfect, hedge.

Finally, we study a continuous-time Brownian environment, as in Brunnermeier and Sannikov (2014) and others, mainly for tractability. We also consider some alternatives. Section 4 explores the setting with Poisson shocks, which allows fire sales to be large disaster events. Online Appendix G presents the discrete-time model with time-step Δ and binomial tree uncertainty. There, we show that the discrete-time equations converge to their continuous-time counterparts as $\Delta \to 0$, where the limit is either Brownian or Poisson uncertainty depending on how the binomial tree is structured. (That said, we do not provide an equilibrium existence proof nor construction for time-step $\Delta > 0$.)

[6]It is not obvious a priori that economic dynamics are invariant to whether (i) there a single shock with fundamental volatility $\sigma = 0$; or (ii) there are both sunspot and fundamental shocks, with the latter frictionlessly hedgeable. Case (i) and case (ii) share the property that a non-hedgeable sunspot shock exists, but they differ in how much fundamental risk the economy has. In fact, the equilibrium equations are *not* all literally identical between the two cases. In particular, the interest rate r will differ (and in the case of non-logarithmic utility, optimal consumption hence the capital price function will also differ). While this is minor, because r will not be relevant for any other object in the economy, we do think it is important pedagogically to establish how the financial market structure by itself (without the confounding effect of setting fundamental volatility to zero) affects all the equations.

2 Brownian Sunspot Equilibrium (BSE)

Our main results construct and characterize a Brownian Sunspot Equilibrium (BSE).

2.1 BSE: existence, uniqueness, and properties

To construct a BSE, start from the conjecture that the capital price is only a function of η, i.e., $q_t = q(\eta_t)$ for some function q. By Itô's formula, $\sigma_q = \frac{q'}{q}\sigma_\eta$ and $\varsigma_q = \frac{q'}{q}\varsigma_\eta$. Combining this with equations (17)-(18), we have $\sigma_q = 0$ and

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\varsigma_q = 0. \tag{21}$$

There are two possibilities: either (i) $\varsigma_q = 0$; or (ii) $1 = (\kappa - \eta)\frac{q'}{q}$, in which case ς_q can be non-zero.

Consider first the situation where $\kappa = 1$. It must be that $\varsigma_q = 0$ in this region. Indeed, if not then (21) implies $1 = (1-\eta)\frac{q'}{q}$, which requires q to be an increasing function of η. On the other hand, (PO) implies that $q = a_e/\bar{\rho}(\eta)$ is a decreasing function of η. Thus, no solution can exist; it must be that $\varsigma_q = 0$ instead.

Next, consider the more interesting situation where capital is inefficiently allocated: $\kappa < 1$. In this situation, $\varsigma_q = 0$ cannot hold (to see this, plug $\varsigma_q = 0$ into the risk-balance condition (RB) to see that $\kappa = 1$ would be required). And so equation (21) can only hold if $1 = (\kappa - \eta)\frac{q'}{q}$. Substituting $\kappa = \frac{q\bar{\rho} - a_h}{a_e - a_h}$ from (PO), we obtain a first-order ODE for q:

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1-\eta)a_h}, \quad \text{if} \quad \kappa < 1. \tag{22}$$

Consider boundary condition $\kappa(0) = 0$, equivalently $q(0) = a_h/\rho_h$ by (PO). Our convention throughout the paper is to treat expressions like "$\kappa(0)$" as the limit $\lim_{\eta\to 0}\kappa(\eta)$, since η never literally equals zero in any equilibrium. Intuitively, this boundary condition says that experts fully de-lever as their wealth shrinks. (Section 2.3 below considers other boundary conditions. Later in the paper, we also provide a sense in which $\kappa(0) = 0$ is a natural choice.) Equipped with $\kappa(0) = 0$, ODE (22) is solved on the endogenous region $(0, \eta^*)$ where households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. Given a solution for (q, κ), the risk-balance equation (RB) yields capital price variance

$$\varsigma_q^2 = \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1. \tag{23}$$

Since $\varsigma_q \neq 0$ in (23), a BSE exists as long as ODE (22) has a solution. Unfortunately, the singularity $\lim_{\eta\to 0} q'(\eta) = \lim_{\eta\to 0} q(\eta)(\kappa(\eta)-\eta)^{-1} = +\infty$ forces us to go beyond standard ODE existence/uniqueness results. Instead, we build a monotonic sequence of auxiliary economies that converge to the BSE.

Proposition 1 (BSE). *There exists a unique BSE with $\kappa(0) = 0$. In this BSE, $\varsigma_q(\eta) \neq 0$ on $(0, \eta^*)$ for $\eta^* > 0$; $\varsigma_q(\eta) = 0$ on $(\eta^*, 1)$; and the process $(\eta_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution on $(0, \eta^*]$.*

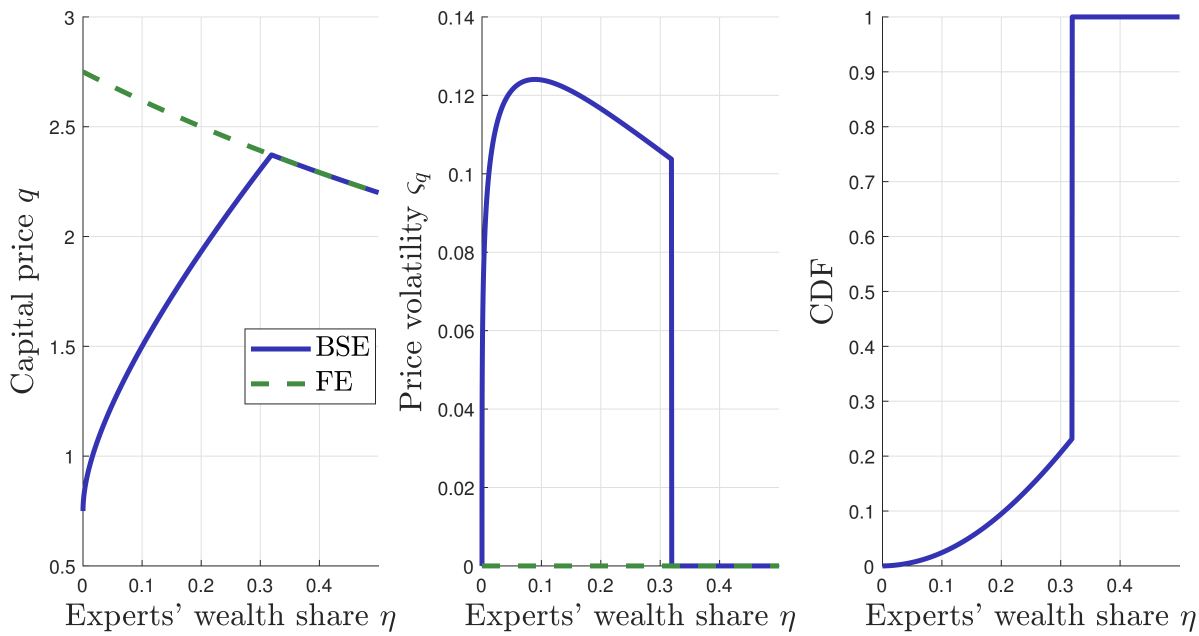

Figure 1: Capital price q, volatility ς_q, and stationary CDF of η in the Brownian Sunspot Equilibrium (BSE) and Fundamental Equilibrium (FE). The stationary CDF is calculated using the Kolmogorov Forward Equation for η. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$. (Note that g and σ are irrelevant to the solution.)

Figure 1 displays a numerical example with the capital price q and volatility ς_q as functions of η.[7] The left region where q is upward sloping corresponds to the inefficient region where $\kappa < 1$ (i.e., $\eta < \eta^*$). This region induces a non-trivial amount of volatility (middle panel). Volatility can be so high because of the large productivity gap $a_e - a_h$; this gap makes fire sales impact asset prices significantly. By contrast, when capital is efficiently allocated ($\kappa = 1$, or equivalently $\eta \geq \eta^*$), the economy behaves exactly as in the Fundamental Equilibrium, and endogenous price volatility is zero. Finally,

[7] The numerical procedure for obtaining a BSE is as follows. Equation (22) is a first-order ODE. Thus, starting from the initial condition $q(0) = a_h/\rho_h$, we may use (22) to numerically obtain a candidate function $\tilde{q}(\eta)$ for all $\eta > 0$. Let η^* be the first point where $\tilde{q}(\eta)$ exceeds $a_e/\bar{\rho}(\eta)$ (equivalently, the associated capital share $\tilde{\kappa}(\eta)$ exceeds 1). Then, a solution is defined as $q(\eta) = \tilde{q}(\eta)$ for $\eta \leq \eta^*$ and $q(\eta) = a_e/\bar{\rho}(\eta)$ for $\eta > \eta^*$. By uniqueness, this is the only BSE solution with boundary condition $\kappa(0) = 0$.

notice that the equilibrium is stationary (the right panel plots the stationary CDF of η). This is fairly easy to understand. Since $\rho_e > \rho_h$, experts consume at a higher rate than households, so they will never control all wealth in the economy. On the other hand, expert wealth will not vanish because they hold a disproportionate share of capital, which delivers a risk premium in equilibrium due to the volatility ς_q.

2.2 Intuition behind self-fulfilling fire-sales

A BSE is a self-fulfilling fire sale. The mechanics are as follows. If agents believe sunspots can affect asset prices, then the actual arrival of such a shock triggers trading of capital between experts and households. Why? While experts are able to share risks from the fundamental shock dW, they cannot hedge the sunspot dZ. Anticipating a hit to their balance sheet from a decline in capital valuations, experts rush to sell capital.

Let us explore further. To sustain a self-fulfilling fire sale, it must be that (i) asset prices fall, and (ii) the asset price decline disproportionately reduces expert wealth. Then, the self-fulfillment mechanism enters: (iii) the disproportionate decline in expert wealth justifies selling behavior. Mathematically, since we are studying an equilibrium that is Markovian in η, hence looking for a function $q(\eta)$, the core intuition is transparent: the key question is whether or not some shock can trigger a decline in q (step i), which then affects η (step ii), which then feeds back into q (step iii).

First, why do asset prices fall? Asset prices are connected to capital holdings because experts are more productive than households hence willing to pay more for capital. Equation (PO) captures this idea via the positive relationship between q and κ. Thus, if all experts coordinate to sell capital, its price will fall:

$$\kappa \downarrow \quad \Longrightarrow \quad q \downarrow \tag{H1}$$

The key assumption underlying this mechanism is productivity heterogeneity, $a_e > a_h$.

Second, why does experts' relative wealth fall? In general, falling asset prices do not necessarily damage expert balance sheets; for instance, if asset prices were hypothetically to fall for fundamental reasons, this shock would be perfectly shared between experts and households. Such a proportionate wealth response would in fact eliminate the possibility of a self-fulfilling fire sale. But if asset prices decline due to a non-hedgeable sunspot shock, experts' have disproportionate exposure to this shock and their relative wealth would fall—see equation (18). Thus,

$$q \downarrow \quad \Longrightarrow \quad \eta \downarrow \quad \text{(if the shock is non-fundamental)} \tag{H2}$$

The key assumption underlying this mechanism is the market incompleteness for hedging the sunspot shock dZ. Putting together (H1)-(H2), a coordinated fire-sale does affect experts' balance sheets.

The final question is how the mechanism of self-fulfillment arises. Beliefs are critical: if agents believe other agents will participate in a fire sale, then they will do so as well. Agents must decide how they think lower expert wealth feeds back into trading behavior. Mathematically, since we study Markovian equilibria in η, the entire question boils down to whether agents believe the decline in η from (H2) feeds back into κ:

$$\eta \downarrow \quad \overset{?}{\Longrightarrow} \quad \kappa \downarrow \tag{H3}$$

There are two possibilities so far. On the one hand, agents may think that other experts' wealth is disconnected from their trading behavior, so that (H3) is inoperative. In that case, a self-fulfilling fire sale cannot be justified. The disconnect between expert wealth and trading behavior happens both when η is very high (so experts have sufficient wealth to endure shocks without selling) and if agents are playing the safe Fundamental Equilibrium (FE). On the other hand, agents may suspect that other experts will sell when they are undercapitalized. In that case, putting together (H1)-(H3) leads to the dynamic feedback loop:

$$\kappa \downarrow \quad \Longrightarrow \quad q \downarrow \quad \Longrightarrow \quad \eta \downarrow \quad \Longrightarrow \quad \kappa \downarrow \tag{H-loop}$$

The feedback loop in (H-loop) captures the idea that a perceived fire sale justifies itself, intermediated by expert balance sheets.

When does the feedback loop in (H-loop) terminate? Put differently, how large will the fire sale be? Here, it helps to distinguish a relative and absolute effect. The co-dependence of q and η is captured by the pricing function $q(\eta)$, which only captures *relative* responses: how much asset prices move per unit of wealth share fluctuation. Such relative responses are pinned down by ensuring consistency of $\varsigma_\eta = (\kappa - \eta)\varsigma_q$ with $\varsigma_q = \frac{q'}{q}\varsigma_\eta$, leading to the ODE (22) for $q(\cdot)$. Clearly, these two equations could admit any scaling of ς_q by also scaling ς_η, reinforcing the notion that they only pin down relative responses. The *absolute* size of the fire sale is, instead, pinned down by the risk-balance condition (RB). If both experts and households are marginal in capital markets, their relative risk exposure $\frac{\kappa}{\eta}\varsigma_q^2 - \frac{1-\kappa}{1-\eta}\varsigma_q^2$ must be matched by their relative expected returns $\frac{a_e}{q} - \frac{a_h}{q}$. This pins down ς_q, the absolute effect of the non-fundamental shock. In other words, the size of an ex-post fire sale is pinned down by agents' ex-ante optimal capital holdings and their rational expectation of the fire sale.

Having explained the mechanism behind self-fulfilling fire sales, we now address a few auxiliary issues. First, our mechanism is critically dynamic and uncertainty-driven. Within the sunspot literature, our paper is closest to the literature on banking panics—with "run" and "no-run" equilibria being analogues to our BSE and FE. Unlike this run literature, our BSE requires an infinite horizon because fire sales rely on future uncertainty in asset prices. In a finite-horizon version of our economy, the asset price q_t would be pinned down at some future date T, rendering it risk-free at all earlier dates by backward induction. But if asset prices are riskless, the economy must be in the FE. (This can be seen by plugging in $\varsigma_q = 0$ in the risk-balance equation (RB).)

Second, unlike the literature on runs, there are actually many self-fulfilling fire-sale equilibria. Section 2.3 document these and shows that they are determined by coordination on the boundary condition $\kappa_0 := \lim_{\eta \to 0} \kappa(\eta)$. So far, we have studied the BSE with $\kappa_0 = 0$, which corresponds to the maximal fire sale when experts are impoverished. But there are also a continuum of "partial fire-sale" equilibria, associated to $\kappa_0 \in (0, 1)$.

Finally, while our sunspot analysis is simplest and cleanest in the case of log utility and Brownian shocks, the intuition described above suggests that nothing is special about log nor Brownian motion. To confirm this, we perform two extensions. First, Online Appendix D shows how BSEs can be obtained with CRRA preferences and risk aversion $\gamma \neq 1$. Second, Section 4 below illustrates a sunspot equilibrium with Poisson jumps instead of Brownian shocks.

2.3 Beliefs about disaster states

In this section, we outline a richer class of BSEs. The entire set of BSEs studied here will be indexed by agents' beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Mathematically, recall that we previously have assumed the boundary condition $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. Strictly speaking, $\kappa(0) = 0$ turns out to not be necessary without some equilibrium refinements, and it will be interesting to relax this assumption.

Consider any $\kappa_0 \in [0, 1)$ and enforce $\lim_{\eta \to 0} \kappa(\eta) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is similar to Proposition 1, with the generalization that the boundary condition to the ODE (22) is now $\kappa(0) = \kappa_0$. Along the way toward proving Proposition 1, we actually proved that there is a unique solution to this ODE, hence a unique sunspot equilibrium for each $\kappa_0 < 1$. For short, let us refer to these equilibria as BSE(κ_0).

Corollary 1 (Disaster beliefs)**.** *For each $\kappa_0 \in [0,1)$, there exists a unique BSE(κ_0) with $\kappa(0) = \kappa_0$, in which $\varsigma_q(\eta) \neq 0$ on $(0, \eta^*_{\kappa_0})$ for $\eta^*_{\kappa_0} > 0$, and $\varsigma_q(\eta) = 0$ on $(\eta^*_{\kappa_0}, 1)$. Moreover, the process $(\eta_t)_{t \geq 0}$ possesses a non-degenerate stationary distribution on $(0, \eta^*_{\kappa_0}]$.*

In the proof of Proposition 1, we also showed that our baseline BSE(0) is the result of taking the limit $\kappa_0 \to 0$ in the BSE(κ_0). Similarly, one can show that as $\kappa_0 \to 1$, the BSE(κ_0) converges to the FE. For any $\kappa_0 \in (0,1)$, an intermediate sunspot equilibrium can prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this simple way, the boundary condition $\kappa_0 \in [0,1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure 2.

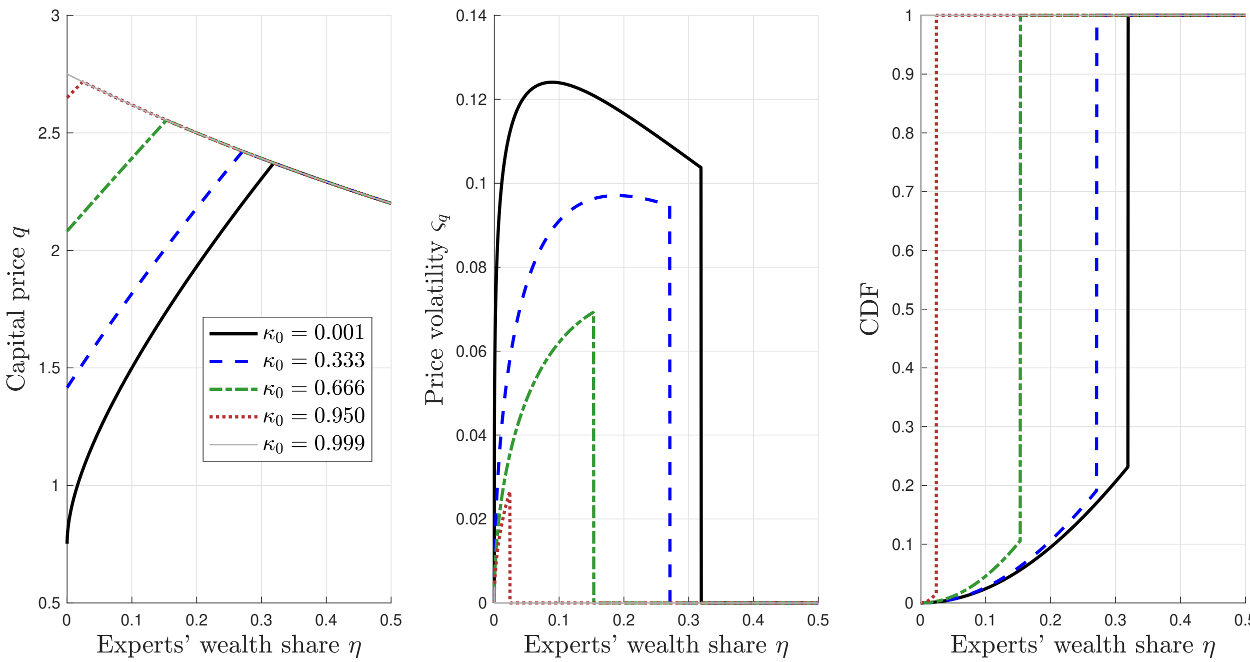

Figure 2: Capital price q, sunspot volatility ς_q, and stationary CDFs of η for different levels of disaster belief κ_0. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$.

The indeterminacy in κ_0 reinforces of the central property that the degree of fire sales is indeterminate in these models. Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales in response to sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism. At this point, we have a single Fundamental Equilibrium (which one may think of as $\kappa_0 = 1$) and a continuum of Brownian Sunspot Equilibria (with any $\kappa_0 < 1$). How do these equilibria compare to conventional financial accelerator equilibria where fundamental shocks cannot be hedged? Is there any reason to think one of these κ_0 is a more natural outcome than the others? We turn to these questions next.

3 Self-fulfilling fire sales as a limiting case

3.1 Observational equivalence to conventional accelerator equilibria

So far, we have allowed agents to perfectly hedge fundamental shocks. To compare our results to the literature, and provide further interpretation, consider what happens if no market existed for hedging dW_t. Equilibria for this "conventional situation" are studied extensively, with the defining feature that fundamental shocks are amplified by endogenous wealth dynamics.

Definition 3. A *Conventional Accelerator Equilibrium* (CAE) is an equilibrium which satisfies Definition 1 with the additional requirement that $x_{e,t} = x_{h,t} = 0$ (no hedging), which is Markovian in the expert wealth share η, and which satisfies $\sigma + \sigma_q > 0$.

Let us briefly recount the details for a CAE in which dW_t is non-hedgeable and the sunspot shock dZ_t is absent. (In fact, Lemma A.2 shows that if dW_t is non-hedgeable, then dZ_t cannot impact equilibrium.) The key modification is that non-hedgeable return volatility is now $\sigma + \sigma_q$ rather than ς_q from the BSE. Thus, the entire set of equilibrium equations is similar to before, except ς_q is replaced by $\sigma + \sigma_q$ and ς_η by σ_η in all cases. (There is also a small modification to the expression for r. All the CAE equations are contained in Appendix A.1.) The qualifier in Definition 3 that $\sigma + \sigma_q > 0$ is to focus on the standard solution where capital returns respond in the same direction as the shock.[8]

Follow a similar analysis that led to the critical equation (21). Solving the two-way feedback between the Itô condition $\sigma_q = \frac{q'}{q}\sigma_\eta$ and wealth volatility $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$, we obtain

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{24}$$

Equation (24) is often interpreted as *amplification*, because $\frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa - \eta)\sigma$, which reduces asset prices through q'/q. This explains the numerator of (24). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa - \eta)$, which feeds back into a $(\kappa - \eta)q'/q$ percent further reduction capital prices, which then triggers the

[8] Online Appendix C.1 presents a solution where $\sigma + \sigma_q < 0$ in the "fire sale region," confirming a conjecture in footnote 16 of Kiyotaki and Moore (1997). This counterintuitive equilibrium can arise because only the return variance $(\sigma + \sigma_q)^2$ is pinned down by the risk-balance condition (A.14). Thus, the sign of $\sigma + \sigma_q$ can be positive or negative.

loop again. The second-round impact is $[(\kappa - \eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa - \eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

In the BSEs, recall that $(\kappa - \eta)q'/q = 1$ (equation (21)). BSEs have no dampening in successive rounds of the feedback loop, leading to infinite amplification! Despite this contrast, it turns out that the baseline BSE is "close" to a conventional equilibrium. As σ shrinks, amplification rises because falling exogenous volatility incentivizes expert leverage, which raises endogenous volatility. As σ vanishes, amplification rises explosively and equilibria become sunspot-like.[9]

Proposition 2 (Observational equivalence)**.** *Suppose a CAE exists for each $\sigma > 0$ small enough, with $\kappa(0) = 0$. Then, as $\sigma \to 0$, the CAE converges to the BSE($\kappa_0 = 0$) in distribution.*

It is relatively clear that taking $\sigma \to 0$ in equation (24) yields $[1 - (\kappa - \eta)q'/q]\sigma_q = 0$, analogous to the critical equation (21) from the benchmark model. Yet it is not clear why the solution $\sigma_q \equiv 0$ (hence $\kappa \equiv 1$, i.e., the safe Fundamental Equilibrium) is ruled out as a limiting equilibrium. Our formal proof rules this out and shows that fire sales remain non-negligible in the limit.

The observational equivalence result of Proposition 2 formalizes how the BSE of Proposition 1 "looks similar" to the conventional equilibria that have been studied in the financial accelerator literature. There are two take-aways. Theoretically, our finding demonstrates how the self-fulfilling nature of fire sales is core to the economics of the financial accelerator. Practically, our finding can also be viewed as a robustness result: the dynamics of conventional accelerator equilibria are robust to the inclusion of markets for hedging fundamental risks. In that sense, and supposing there is a natural equilibrium selection device that picks the fire-sale equilibrium, the market incompleteness assumptions made in Brunnermeier and Sannikov (2014) are innocuous. We turn to this selection device in the next section.

Rather than taking $\sigma \to 0$, an alternative way to demonstrate the robustness of fire-sale dynamics is to hold σ fixed and take a limit as markets "become complete." Online Appendix F analyzes a particular version of this complete-markets limit. There, we suppose experts can issue up to $1 - \chi$ fraction of their own risks as equity to the market and must retain χ fraction of their risks. We first provide a precise sense in which, for

[9] Brunnermeier and Sannikov (2014) provide a related limiting result, arguing numerically that asset-price volatility does not vanish as $\sigma \to 0$, also known as the "volatility paradox." They also provide an analytical result that $\lim_{\eta \to 0} \frac{\sigma_\eta}{\eta} = \frac{a_e - a_h}{a_h} \frac{\rho_h}{\sigma} + O(\sigma)$. We go further in proving that the entire equilibrium converges, as $\sigma \to 0$, to a sunspot equilibrium. Related results can be found in Manuelli and Peck (1992) and Bacchetta et al. (2012), in which sunspot equilibria could be seen as limits of fundamental equilibria when fundamental uncertainty vanishes.

any $\chi > 0$, the equilibrium pricing function with equity-issuance, once the state variable is re-scaled, is approximately similar to the one without equity-issuance (Proposition F.1). We then show that as $\chi \to 0$, when financial markets approach completeness, the *possibility* of fire-sales does not vanish (Proposition F.2). These points convey a sense in which the fire sales associated to the conventional financial accelerator are robust to allowing a significant amount of risk-sharing. Nevertheless, we also show numerically that the probability of a fire-sale shrinks as χ shrinks, eventually vanishing as $\chi \to 0$ for our example. In this sense, approaching complete markets does actually restore the safety and efficiency of equilibrium, which distinguishes the limit $\chi \to 0$ from $\sigma \to 0$.

Finally, notice that Proposition 2 assumes the boundary condition $\kappa(0) = 0$ for each of the CAEs. Online Appendix C.2 shows that this boundary condition is technically not pinned down and numerically constructs CAEs for various $\kappa(0) \in (0, 1)$, mirroring the results for the BSEs in Section 2.3. In some sense, the literature has picked the worst possible CAE (minimal-price, maximal-volatility) by imposing $\kappa(0) = 0$, so far without any rigorous justification.[10]

We address the plausibility of $\kappa(0) = 0$ with a simple equilibrium refinement in Online Appendix C.3. Assume agents face a leverage constraint when holding capital

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \beta, \tag{25}$$

for some $\beta > 1$. Constraint (25) is motivated in the appendix by a standard limited commitment friction. We prove that, as $\beta \to \infty$ and the constraint becomes arbitrarily lax, the unique outcome is the CAE with $\kappa(0) = 0$ (Proposition C.1). The argument is involved, but intuitively, the leverage constraint gives experts an additional motive to sell capital, which forces coordination on maximal selling in response to negative shocks.

3.2 Equilibrium selection via a small-noise limit

So far, we have demonstrated that BSEs are a *possibility*. But our model inherently permits multiple equilibria. Agents may just as well coordinate on the Fundamental Equilibrium, which has no fire sales, or they may coordinate on a BSE with disaster belief $\kappa_0 > 0$ so that the fire sale is only partial. In this section, we provide a very simple ratio-

[10] Brunnermeier and Sannikov (2014) argue heuristically for $\kappa(0) = 0$ in their online appendix: "because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households." But, as shown in Lemma C.1, even if $\kappa(0) > 0$, the resulting dynamics of η_t will not allow it to ever reach 0, so there is no contradiction to equilibrium. In other words, experts are almost-surely never pushed to their solvency constraint, so they may never need to fully liquidate.

nale for selecting the BSE with $\kappa_0 = 0$ as the *unique equilibrium*. The idea: suppose agents perceive sunspot shocks as having a small fundamental impact. Even as the perceived fundamental impact vanishes, equilibrium requires sunspots to matter.

Suppose agents believe that dZ is a second *fundamental* shock that affects capital. But unlike dW, there are no hedging markets for dZ. Mathematically, introduce parameter ς in the perceived capital evolution:

$$dk_{i,t} = k_{i,t}[gdt + \sigma dW_t + \varsigma dZ_t]. \tag{26}$$

In reality, $\varsigma = 0$ so that dZ is a sunspot that does not affect capital evolution. Think of ς as small, since we eventually take $\varsigma \to 0$ and impose convergence to rational beliefs.[11]

Definition 4. A *Perceived Accelerator Equilibrium* (PAE) is an equilibrium which satisfies Definition 1, in which agents perceive capital dynamics (26), which is Markovian in the expert wealth share η, and which satisfies $\varsigma + \varsigma_q > 0$.

This perceived risk model is tractable because all the equations are either identical to or limiting versions (as $\varsigma \to 0$) of those that arise when the risk is real. (All the equations are contained in Appendix A.3.) Thus, all the relevant equilibrium equations in the PAE converge to the BSE equations as $\varsigma \to 0$.

The only question is which of the BSE solutions, or possibly the FE, is selected. The equilibrium selection argument proceeds in two steps. First, it is straightforward to verify that the argument in Online Appendix C.3 continues to apply to this perceived-risk setting: a vanishingly-small limited-commitment friction prunes all disaster beliefs besides $\kappa(0) = 0$. Second, taking as given that $\kappa(0) = 0$, we take $\varsigma \to 0$ and show that the BSE emerges as the unique limiting equilibrium.

Essentially, all we are doing in this second step is using the previous limiting results in a different way. Indeed, for any $\varsigma > 0$, agents perceive some fundamental risk, and so their behavior and their *perceived dynamics* mimic a conventional equilibrium, the CAE. By the exact same argument as Proposition 2, this behavior and belief converge to those of the BSE as $\varsigma \to 0$. By imposing that belief distortions are "small" (i.e., requiring agents' beliefs to converge to rationality), the *actual dynamics* must coincide asymptotically with perceived dynamics, and so they coincide with the BSE.

[11] Note that, in a diffusion model, misperceptions about volatility are extreme in the sense in that such beliefs are singular with respect to the objective probability—data at infinitely-high frequency could detect the true volatility. That said, we will take misperception $\varsigma \to 0$ in this argument. And so if investors receive data at anything less than infinitely-high frequency, the belief distortion can be interpreted as trivial.

Proposition 3 (Refinement)**.** *Suppose a PAE exists for each $\varsigma > 0$ small enough, with $\kappa(0) = 0$. Suppose agents' beliefs converge to rational expectations as $\varsigma \to 0$. Then, as $\varsigma \to 0$, the PAE converges to the BSE($\kappa_0 = 0$).*

4 "Large" self-fulfilling fire sales

We now re-write the model with jumps rather than Brownian shocks. For simplicity we assume a single Poisson shock dJ_t, which is a fundamental shock if it has a non-zero impact on capital and is a sunspot shock otherwise. We assume there are no markets to hedge the dJ shock. To focus on bad shocks, we assume that the Poisson shock reduces capital and restrict our analysis to equilibria where asset prices decline in response to this shock.

In this section, we will provide an overview of the jump model and present some numerical results. More details and derivations for this Poisson environment are contained in Online Appendix E. There, we also provide an overview of the numerical method, which is more challenging than simply solving an ODE as in the BSE.

Let capital evolve as

$$dk_t = k_{t-}[gdt - \zeta dJ_t],$$

where J is a Poisson process with intensity λ, and where $\zeta \geq 0$. If $\zeta = 0$, then J is a sunspot shock. Capital prices follow a process of the form

$$dq_t = q_{t-}[\mu_{q,t-}dt - \zeta_{q,t-}dJ_t].$$

Note that $-\zeta_{q,t-}dJ_t := \frac{q_t - q_{t-}}{q_{t-}}$ is the proportional price jump. We will assume that $\zeta_{q,t-}$ is pre-determined, i.e., conditional on a jump, the jump size is known. This dramatically simplifies the analysis, although there may be additional equilibria where the jump size is also random. As usual, we will focus attention to equilibria which are Markov in the expert wealth share η, which follows a process of the form

$$d\eta_t = \mu_{\eta,t-}dt - \zeta_{\eta,t-}dJ_t.$$

Note that $-\zeta_{\eta,t-}dJ_t := \eta_t - \eta_{t-}$ by definition. In this context, Markov equilibria are categorized as follows. If $\zeta > 0$, we define a *Conventional Accelerator Equilibrium* (CAE) as an equilibrium in which $\zeta + \zeta_q - \zeta\zeta_q > 0$. If $\zeta = 0$, we define a *Poisson Sunspot Equilibrium* (PSE) as an equilibrium in which ζ_q is not identically zero.

For convenience, let us define the negative jump in return-on-capital

$$\zeta_R := \zeta + \zeta_q - \zeta\zeta_q. \tag{27}$$

This measures the total risk embedded in managing capital, namely the proportional size of a jump to qk. The two assets in the economy are capital and the risk-free bond. Since our jumps have a known size, optimal portfolio conditions are (see Online Appendix E for a derivation)

$$\begin{aligned} \frac{a_e}{q} + g + \mu_q - r &= \frac{\lambda\zeta_R}{1 - \frac{\kappa}{\eta}\zeta_R} \\ \frac{a_h}{q} + g + \mu_q - r &\leq \frac{\lambda\zeta_R}{1 - \frac{1-\kappa}{1-\eta}\zeta_R}, \quad \text{with equality if } \kappa < 1. \end{aligned}$$

Combining these two equations, we obtain a modified risk-balance equation:

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)} \frac{\lambda\zeta_R^2}{\left(1 - \frac{\kappa}{\eta}\zeta_R\right)\left(1 - \frac{1-\kappa}{1-\eta}\zeta_R\right)}\right]. \tag{RBJ}$$

This is very similar to (RB) from the Brownian model, but where the non-hedgeable risk ς_q is replaced by non-hedgeable jump risk $\frac{\lambda\zeta_R^2}{(1-\frac{\kappa}{\eta}\zeta_R)(1-\frac{1-\kappa}{1-\eta}\zeta_R)}$, which involves both the arrival rate λ, the squared jump size ζ_R^2, and its impact on marginal utilities (terms in the denominator).

The wealth share jump ζ_η is derived by using knowledge of the jump size in q and noting that agents' portfolios (capital and bonds) are predetermined:[12]

$$\zeta_\eta = (\kappa - \eta)\frac{\zeta_R}{1 - \zeta_R}. \tag{28}$$

On the other hand, once the post-jump wealth share is known, the capital price is also known, since η is the sole state variable, i.e., we have $q_t = q(\eta_t)$ for some function q.

[12] The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\zeta_\eta = \eta - \hat{\eta} = B[1/(\hat{q}\hat{K}) - 1/(qK)] = qK(\kappa-\eta)[1/(\hat{q}\hat{K}) - 1/(qK)]$. Using the definitions $\zeta := 1 - \hat{K}/K$ and $\zeta_q := 1 - \hat{q}/q$, and using $\zeta_R := \zeta + \zeta_q - \zeta\zeta_q$, we arrive at $\zeta_\eta = (\kappa-\eta)[(1-\zeta_R)^{-1} - 1]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump.

Thus, if we denote the post-jump wealth share by $\hat{\eta} := \eta - \zeta_\eta$,

$$\zeta_q = -\frac{q(\hat{\eta}) - q(\eta)}{q(\eta)}. \tag{29}$$

These equations encode a two-way feedback between the wealth distribution and capital prices, similar to the Brownian model. Indeed, equations (27)-(28) show that wealth share jump depends on the return jump, which depends on the capital price jump. On the other hand, equation (29) shows that the capital price jump depends on the wealth share jump.

Because we do not model bankruptcy procedures, we must also make sure the jump renders experts solvent, meaning $\zeta_{\eta,t-} < \eta_{t-}$, to preserve the risk-free status of the bond. If solvency cannot be ensured, then no jump can take place. Putting these results together, the equations characterizing an equilibrium of this model are given by the following simple lemma.

Lemma 2 (Equilibrium with Jumps)**.** *A Markov equilibrium with jumps requires functions $(q, \kappa, \zeta_q, \zeta_\eta)$ of η to satisfy price-output relation (PO), risk-balance condition (RBJ), equations (28)-(29), and $\zeta_\eta < \eta$.*

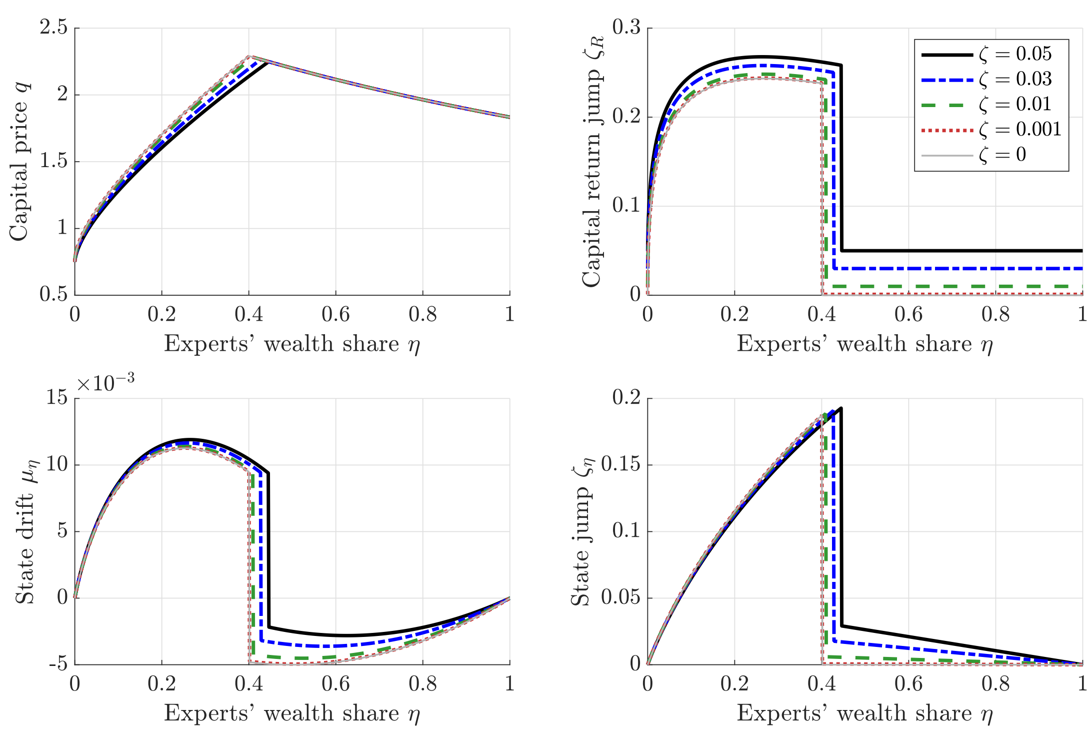

Figure 3: Convergence to a Poisson Sunspot Equilibrium as $\zeta \to 0$. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\lambda = 0.1$. In all cases, we use the boundary condition $\kappa(0) = 0$.

Analogously to the results of Section 3.1, a sunspot equilibrium emerges in the limit as fundamental risk vanishes, $\zeta \to 0$. That is, the CAE converges to the PSE. While we do not prove this rigorously in the jump context, we do provide a numerical example in Figure 3.[13] As exogenous risk shrinks, the pricing function $q(\eta)$ converges to a limit which differs from the riskless Fundamental Equilibrium (FE). While the fire-sale region shrinks with ζ, it does not vanish, and endogenous jump risk remains in the limit. This is despite the fact that when $\zeta = 0$ (i.e., zero exogenous risk), there always exists a safe Fundamental Equilibrium (FE) featuring $\kappa \equiv 1$ and $\zeta_q \equiv 0$ at all times.

Next, we compare experts' capital share κ in the PSE to the Brownian Sunspot Equilibrium (BSE) and the safe Fundamental Equilibrium (FE). Figure 4 displays the results. The left panel plots one simulation of κ in these three equilibria. The right panel plots the stationary densities of κ (although note that the BSE and FE "densities" in fact have a point mass at $\kappa = 1$). This example illustrates how misallocation in the PSE tends to be worse than in the BSE, and certainly much worse than the efficient FE. The reason is simple: the PSE features a very left-skewed distribution of sunspot shocks, hence large fire sales. This means that asset prices and experts' capital share can immediately become quite depressed tomorrow, even if the economy is doing well today.

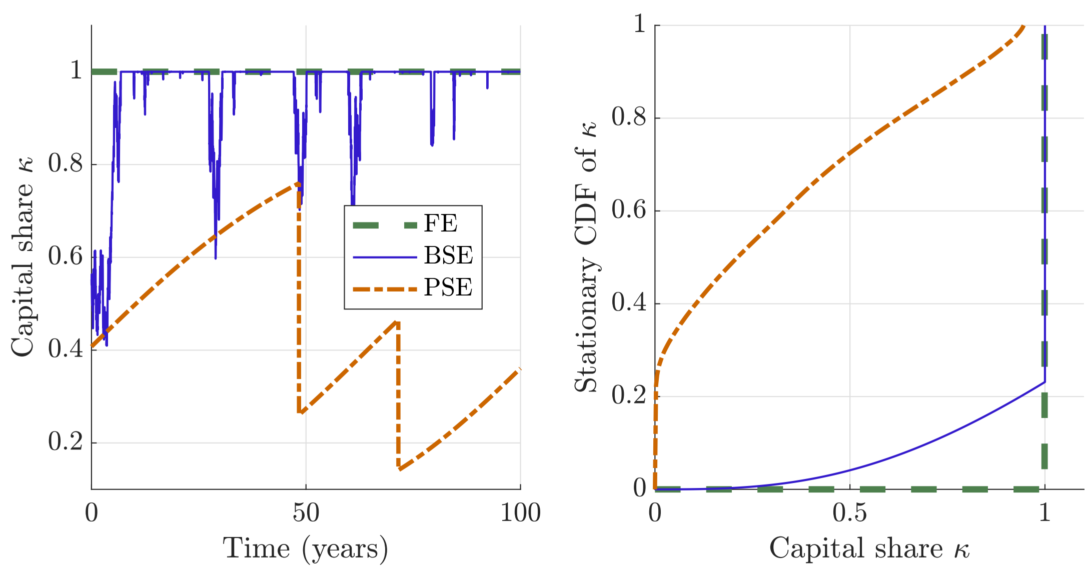

Figure 4: Time series and stationary density of capital price q in a PSE, BSE, and Fundamental Equilibrium (FE). Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0$, $\zeta = 0$. We use the boundary condition $\kappa(0) = 0$ in all cases. For the PSE, we use $\lambda = 0.02$ as the arrival rate of Poisson jumps and compute the stationary CDF via a 100,000 year simulation at the weekly frequency.

[13] Such a convergence proof would involve a much harder existence/uniqueness analysis of differential equations with "delay". That said, one can easily verify that all the equilibrium equations are continuous in ζ, so that the limiting equations as $\zeta \to 0$ are well-defined. For this reason, a reasonable conjecture is that the PSE emerges in the limit $\zeta \to 0$.

Overall, we make two points in light of these results. First, different sunspot equilibria are possible depending on what agents assume about the sunspot process. Agents can equally well coordinate on large, discrete fire sales as they can on small, frequent trading. Second, whether we model shocks as Brownian motions or Poisson jumps, sunspot equilibria are "close" to their corresponding conventional accelerator equilibrium. In that sense, the traditional assumption of non-hedgeable fundamental shocks may be relatively innocuous to the equilibrium dynamics.

5 Conclusion

We have studied a canonical macro-finance model and constructed an equilibrium with self-fulfilling fire sales. The key innovation is that, while all fundamental risks are perfectly shared, not every conceivable shock is hedgeable. Fundamentals-based fire sales are no longer possible, but endogenously-emerging risks are unhedgeable and could matter. If agents coordinate on selling capital, its price falls, which feeds back into net worth and self-justifies the initial fire sale. The resulting dynamics are familiar, resembling the conventional financial accelerator equilibria in a sense we make precise, but can only emerge out of non-fundamental shocks. For example, consider the emergence of new types of shocks for which hedging markets have not yet developed; these are the shocks likely to encourage coordination and fire sale behavior. Finally, despite the presence of multiple equilibria, we provide a simple trembling-hand-style refinement, based on a vanishingly-small limited commitment problem, combined with agents mistaking the sunspot shock to have a vanishingly-small fundamental impact. This refinement justifies selecting the fire-sale equilibrium and neglecting the safe equilibrium.

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Appendix to: Dynamic Self-Fulfilling Fire Sales

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February 13, 2025

A Proofs

Derivation of mean-variance problem. Here, we briefly justify the mean-variance problem (8). This standard result is due to the combination of log utility and a homogeneous of degree one budget set (including the fact that the no-shorting constraint $k \geq 0$ is homogeneous in agent-specific wealth). See Cvitanić and Karatzas (1992) for a full mathematical proof using convex duality methods. Rather than provide the formal proof along those lines, we provide below a sketch of the argument behind the mean-variance problem using the recursive formulation of agents' problems.

Consider an agent with net worth n, who faces the aggregate state vector X (in our equilibria, $X = \eta$). The evolution of n is given by (2), while the evolution of the aggregate state is independent of individual states and controls, i.e., $dX = \mu(X)dt + \sigma(X)dW_t + \varsigma(X)dZ_t$. Letting V denote his value function, the associated HJB equation is

$$\rho V(n, X) = \max_{c \geq 0, k \geq 0, x \in \mathbb{R}} \log(c) + \frac{\mathbb{E}[dV(n, X)]}{dt}$$

We now guess and verify that $V(n, X) = \rho^{-1} \log(n) + \xi(X)$, where ξ is a function solely of aggregate states. Using the conjecture, and using Itô's formula to write $\mathbb{E}[d \log(n)] = \mathbb{E}[\frac{dn}{n}] - \frac{1}{2}\mathrm{Var}[\frac{dn}{n}]$, we can write the HJB equation as

$$\log(n) + \rho\xi(X) = \max_{c, \geq 0, k \geq 0, x} \log(c) + \frac{\rho^{-1}}{dt} \underbrace{\Big(\mathbb{E}[\frac{dn}{n}] - \frac{1}{2}\mathrm{Var}[\frac{dn}{n}]\Big)}_{\text{mean-variance component}} + \frac{\mathbb{E}[d\xi(X)]}{dt}$$

Consumption only appears in the flow utility $\log(c)$ and the drift of dn in (2), and so maximizing over c gives $c = \rho n$. Capital k and hedges x only appear in the net worth dynamics, leading to the mean-variance problem (8). Finally, plugging the optimal consumption, capital, and hedges back into the HJB equation, we can show that the individual state n drops from the expression and the remainder is exclusively a function of X, representing a functional equation for $\xi(X)$. This verifies the conjectured form of the value function, and therefore verifies the conjectured mean-variance problem.

Proof of Lemma 1. The construction is contained in the text leading up to the Lemma. To confirm that this is an equilibrium, note that all equations are satisfied: (PO) by $\kappa = 1$ and $q = a_e/\bar{\rho}(\eta)$; (RB) by $\kappa = 1$ and $\varsigma_q = 0$; and r and π can be set by (13) and (14), respectively. Finally, the dynamics in (19)-(20) are consistent with equations (16)-(18) and Itô's formula applied to $q = a_e/\bar{\rho}(\eta)$. $\square$

Proof of Proposition 1. As stated in the text, the existence of a BSE boils down to proving the existence of a solution q to ODE (22) with boundary condition $\kappa(0) = 0$, or equivalently $q(0) = a_h/\rho_h$. However, because $\kappa(0) = 0$ implies that $q'(0) = +\infty$, we cannot apply standard results to this problem and must argue differently.

In our first step, we replace the boundary condition $\kappa(0) = 0$ by any $\kappa(0) = \kappa_0 \in (0,1)$. We prove existence of a solution to (22) with this modified boundary. In our second step, we take the limit $\kappa_0 \to 0$ and argue the limit satisfies the relevant equations. Our third step shows this limit is the unique solution to the BSE ODE.

Step 1: Existence (and uniqueness) for $\kappa_0 \in (0,1)$. Consider the initial value problem

$$q' = F(\eta, q) := \frac{a_e - a_h}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h} q \quad \text{s.t.} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}. \tag{A.1}$$

Notice that $q'(0+) = \frac{q(0)}{\kappa_0}$ is bounded, ensuring F is bounded and uniformly Lipschitz on the domain $\mathcal{R}_\epsilon := \{(\eta, q) : 0 < \eta < 1, \ (\epsilon + \eta)a_e + (1-\epsilon-\eta)a_h < q\bar{\rho}(\eta)\}$, for each $\epsilon \in (0, \kappa_0)$. This is the relevant domain because $\kappa'(0+) = \frac{\rho_e - \rho_h}{a_e - a_h} q(0) + \frac{\rho_h}{a_e - a_h} q'(0+) = 1 + \frac{a_h}{\kappa_0} + (\kappa_0 + a_h)(\frac{\rho_e - \rho_h}{\rho_h}) > 1$, so that the solution points into the interior of this region as long as $\epsilon \leq \kappa_0$. Thus, the Picard-Lindelöf theorem implies that there exists a *unique* solution q^* to this initial value problem, for $\eta \in (0, b)$, some b. Standard continuation arguments can be used to extend the solution to either the entire domain $\mathcal{R} := \cup \mathcal{R}_\epsilon$ or until a point such that the either solution or F explodes. In other words, either:

(i) $b = 1$; (ii) $q^*(\eta) \to +\infty$ as $\eta \to b$; (iii) b satisfies $ba_e + (1-b)a_h = q^*(b)\bar{\rho}(b)$.

Let us first rule out case (iii). Consider the pricing function $\underline{q}(\eta) = \frac{\kappa_0(a_e - a_h) + \eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}$, which corresponds by equation (PO) to the expert capital share $\underline{\kappa}(\eta) = \kappa_0 + \eta$. Note that $\underline{q}$ uniquely solves the alternative ODE

$$q' = \underline{F}(\eta, q) := \frac{a_e - a_h - (\rho_e - \rho_h)q}{\bar{\rho}(\eta)} \quad \text{s.t.} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}.$$

Since $q^*(0) = \underline{q}(0)$ and since $F(\eta, q) > \underline{F}(\eta, q)$ on $\mathcal{R}$, the comparison theorem for ODEs implies that $q^*(\eta) > \underline{q}(\eta)$ for all $\eta > 0$. Because $\underline{q}(\eta) > \frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}$, this proves that there cannot exist any b with $q^*(b) = \frac{ba_e + (1-b)a_h}{\bar{\rho}(b)}$. In passing, also note that this proves that the solution $q^*(\eta)$ is necessarily such that the associated capital share $\kappa^*(\eta) = \frac{q^*(\eta)\bar{\rho}(\eta) - a_h}{a_e - a_h}$ from equation (PO) is such that $\kappa^*(\eta) > \eta$.

We are left with cases (i) or (ii). In either case, set

$$\eta^* = \inf\{\eta \in (0, b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$$

with the convention that $\eta^* = 1$ if the set is empty. Note that $\eta^* > 0$ is immediate, since $q^*(0) = a_h/\rho_h < a_e/\rho_h$ and since $\frac{d}{d\eta}q^*(0+) = \frac{q^*(0)}{\kappa_0}$ is bounded.

In case (ii), with $b < 1$ and $q^*(b-) = +\infty$, it is clear by the continuity of the solution q^* that $\eta^* < b < 1$.

In case (i) with $b = 1$, we may easily show by contradiction that $\eta^* < 1$. Indeed, if $\eta^* \geq 1$, then $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$. But by continuity of q^*, the only way F could have changed signs is that there exists an $\eta^\circ \in (0, 1)$ such that $\eta^\circ a_e + (1 - \eta^\circ)a_h = q^*(\eta^\circ)\bar{\rho}(\eta^\circ)$. This latter possibility was just ruled out (case (iii)). And so $\eta^* < 1$.

Consequently, in cases (i)-(ii), there exists $0 < \eta^* < 1$ such that $q^*(\eta^*) = a_e/\bar{\rho}(\eta^*)$.

Finally, define

$$q(\eta) := \begin{cases} q^*(\eta), & \text{if } \eta < \eta^*; \\ a_e/\bar{\rho}(\eta), & \text{if } \eta \geq \eta^*. \end{cases}$$

This function satisfies $q' = F(\eta, q)$ on $(0, \eta^*)$, with boundary values $q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}$ and $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$. Thus, we have found a solution to the capital price satisfying all the desired relations. And as shown above, the capital share satisfies $\kappa(\eta) > \eta$.

Equation (23), plus the fact that $\kappa > \eta$, implies $\varsigma_q^2 > 0$ on $(0, \eta^*)$. Since $\eta^* > 0$, we thus have $\varsigma_q(\eta) \neq 0$ on a positive measure subset as desired.

Step 2: Limit as $\kappa_0 \to 0$. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta^*_{\kappa_0})$ be the associated equilibrium capital price and fire-sale threshold. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x))dx, \quad \eta < \eta^*_{\kappa_0}. \tag{A.2}$$

We first claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0, for each x. Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0. By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some

$\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0, we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x. The monotonicity of q_{κ_0} in κ_0 also proves that $\eta^*_{\kappa_0}$, by its definition, is weakly decreasing in κ_0.

Because of these monotonicity results, the following limit $(q_0, \eta^*_0) := \lim_{\kappa_0 \to 0}(q_{\kappa_0}, \eta^*_{\kappa_0})$ exists. This limit is our candidate solution for the BSE. It suffices to show that q_0 satisfies (a) $q_0' = F(\eta, q_0)$ on $(0, \eta^*_0)$, (b) $q_0(0) = a_h/\rho_h$, and (c) $q_0(\eta^*_0) = a_e/\bar{\rho}(\eta^*_0)$.

Combine the monotonicity result for $q_{\kappa_0}(x)$ with the fact that $\partial_q F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0,1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0, for each x. Thus, applying the monotone convergence theorem to (A.2), and recalling that $\eta^*_0 \geq \eta^*_{\kappa_0}$, we have

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x))dx, \quad \eta < \eta^*_0$$

which proves (a), by differentiating, and (b), by substituting $\eta = 0$.

To prove (c), note that q_{κ_0} is a bounded, continuous function for each κ_0. Furthermore, q_{κ_0} converges to q_0 uniformly (i.e., in the sup-norm), due to the fact that $\partial_q F < 0$.[14] Because the space of bounded, continuous functions (equipped with the sup-norm) is a Banach space, it holds that $q_0(x)$ is also a bounded, continuous function. Therefore,

$$q_0(\eta^*_0) = \lim_{n \to 0} q_n\Big(\lim_{m \to 0} \eta^*_m\Big) = \lim_{n \to 0}\lim_{m \to 0} q_n(\eta^*_m) = \lim_{\kappa_0 \to 0} q_{\kappa_0}(\eta^*_{\kappa_0}) = \lim_{\kappa_0 \to 0} \frac{a_e}{\bar{\rho}(\eta^*_{\kappa_0})} = \frac{a_e}{\bar{\rho}(\eta^*_0)}$$

which proves (c).

Step 3: Uniqueness. Suppose two solutions q and $\tilde{q}$ solved the ODE (22) with boundary conditions $\kappa(0) = \tilde{\kappa}(0) = 0$. Let η^* and $\tilde{\eta}^*$ denote the points where $\kappa(\eta)$ and $\tilde{\kappa}(\eta)$ reach 1. By Lemma A.1, it must be the case that $\kappa(\eta) = 1$ on $[\eta^*, 1]$ and $\tilde{\kappa}(\eta) = 1$ on $[\tilde{\eta}^*, 1]$, so it suffices to consider the fire sale regions $(0, \eta^*)$ and $(0, \tilde{\eta}^*)$. Without loss of generality, we may consider the situation $\tilde{q}(\eta) > q(\eta)$ for all $\eta < \eta^\circ$. The reason: if the two solutions ever crossed at some value of η°, then they would necessarily coincide for all $\eta \geq \eta^\circ$.

Since $q(0) = \tilde{q}(0)$, we have

$$q(\eta) - \tilde{q}(\eta) = \int_0^\eta \Big[F(x, q(x)) - F(x, \tilde{q}(x))\Big]dx, \quad \eta < \eta^\circ$$

Recall that F is decreasing in its second argument. Therefore, $\tilde{q} > q$ on $(0, \eta^\circ)$ implies

[14] Indeed, differentiate (A.2) with respect to κ_0 and η to see that $\partial_{\eta\kappa_0} q_{\kappa_0}(\eta) < 0$, and in particular $\partial_{\kappa_0} q_{\kappa_0}(\eta) \leq \partial_{\kappa_0} q_{\kappa_0}(0) = \frac{a_e - a_h}{\rho_h}$. Thus, the convergence rate of $q_{\kappa_0} \to q_0$ is bounded by the rate that $\kappa_0 \to 0$.

$F(x, q(x)) > F(x, \tilde{q}(x))$ for $x < \eta^\circ$, which from the equation above implies $q(\eta) > \tilde{q}(\eta)$, a contradiction.

Step 4: Stationary distribution. The entire argument above holds for $\eta > 0$. Hence, for this argument to constitute a valid equilibrium construction, it must be the case that $(\eta_t)_{t \geq 0}$ never reaches 0, with probability 1. This result, along with the proof that $(\eta_t)_{t \geq 0}$ has a non-degenerate stationary distribution on $(0, \eta^*]$ is presented in Lemma B.1. In fact, there we prove these results for any boundary condition $\kappa_0 \in [0, 1)$. Hence, Corollary 1 is proved as well. □

Lemma A.1. *Consider a BSE with $\kappa(0) = 0$. Then, there exists a threshold $\eta^* > 0$ such that $\kappa(\eta) < 1$ for all $\eta < \eta^*$ and $\kappa(\eta) = 1$ for all $\eta \geq \eta^*$.*

Proof of Lemma A.1. First, note that ODE (A.1) immediately implies $q' > 0$ on the set $\{\eta : \kappa < 1\}$. Next, we prove that $\{\eta : \kappa < 1\} = (0, \eta^*)$ for some η^*. Suppose $\{\eta : \kappa < 1\}$ were not a connected set. Then, there would exist $\eta_2 > \eta_1$ such that $q(\eta_2) < a_e / \bar{\rho}(\eta_2)$ while $q(\eta_1) = a_e / \bar{\rho}(\eta_1)$. But since $\bar{\rho}(\eta)$ is an increasing function, we have

$$q(\eta_2) < a_e / \bar{\rho}(\eta_2) < a_e / \bar{\rho}(\eta_1) = q(\eta_1).$$

This implies that $q' < 0$ for some $\eta \in (\eta_1, \eta_2) \cap \{\kappa < 1\}$, which is a contradiction. This proves that $\{\eta : \kappa < 1\}$ must be an interval. By Step 1 of the proof of Proposition 1, $\{\eta : \kappa < 1\}$ includes $(0, \eta^*)$ for some $\eta^* > 0$. Hence, $\{\eta : \kappa < 1\} = (0, \eta^*)$. □

Proof of Proposition 2. The equations for the CAE are collected in Appendix A.1 below. Due to Lemma A.2, we must plug $\varsigma_q = \varsigma_\eta = 0$ into those equations.

Using Lemma A.3, we obtain the ODE (A.16) for q, which holds on $\{\eta : \eta < \kappa(\eta) < 1\}$. The proposition's stated assumption $\kappa(0) = 0$ serves as a boundary condition for this ODE. For each $\sigma > 0$ small enough, let q_σ denote a solution to this ODE (which exists by assumption). By Lemma A.7 below, this solution q_σ, if it exists, is unique.

The remainder of the proof proceeds as follows. We first prove that, for any $\sigma > 0$, fire sales happen and so ODE (A.16) applies in some region. We then guess-and-verify that fire sales remain in the limit $\sigma \to 0$. Under the guess that the fire sale region does not vanish, we prove that the limiting equilibrium is the BSE. Given this result, we can then verify that the fire sale region does not vanish as $\sigma \to 0$.

Step 1: Fire sales occur for any $\sigma > 0$. By Lemma A.6 below, there is a threshold $\eta^*_\sigma > 0$ such that the unique solution satisfies $\kappa_\sigma < 1$ on $(0, \eta^*_\sigma)$ and $\kappa_\sigma = 1$ on $[\eta^*_\sigma, 1]$. Based on the result that η^*_σ is positive, fire sales happen for any $\sigma > 0$.

Step 2: The limiting equilibrium is the BSE (if fire sales continue in the limit). First, we establish that the relevant limits exist. Note that the ODE generator F_σ in (A.16) is decreasing in σ uniformly, which implies that the solution q_σ is monotonically (weakly) decreasing in σ. By the monotone convergence theorem, the limit $q_0 := \lim_{\sigma\to 0} q_\sigma$ exists, and by association $\eta^*_0 := \lim_{\sigma\to 0} \eta^*_\sigma$ exists. We will guess (and verify in Step 3) that $\eta^*_0 > 0$.

Next, because q_σ is decreasing in σ, implying η^*_σ is increasing in σ, we have $\eta^*_0 = \inf_\sigma \eta^*_\sigma$. Thus, the entire family $(q_\sigma)_{\sigma>0}$ of solutions satisfy

$$q_\sigma(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F_\sigma(x, q_\sigma(x))dx, \quad \eta < \eta^*_0, \tag{A.3}$$

i.e., each q_σ solves its ODE in the smallest interval $(0, \eta^*_0)$. We claim that $\sigma \to 0$ implies

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F_0(x, q_0(x))dx, \quad \eta < \eta^*_0 \tag{A.4}$$

Define $f_\sigma(x) := F_\sigma(x, q_\sigma(x))$. Our first aim is to show that $f_\sigma(x) \to f_0(x)$ pointwise. Our second aim is to show that the functions $(f_\sigma)_{\sigma\in(0,\bar\sigma)}$ are uniformly integrable (UI), for some maximal volatility level $\bar\sigma$ small enough. These two claims then imply that $\int f_\sigma \to \int f_0$, as desired.

First, we prove the claim that $f_\sigma(x) \to f_0(x)$ pointwise on $\{x > 0\}$. Fix $x > 0$, and define the function $\phi(\sigma, q) := F_\sigma(x, q)$ for this x. We restrict the domain of this function to $[0, \bar\sigma] \times \mathcal{Q}(x)$, where $\mathcal{Q}(x) := [q_{\bar\sigma}(x), a_e/\bar\rho(x)]$. Since $q_{\bar\sigma}(x) > (xa_e + (1-x)a_h)/\bar\rho(x)$, we have that $\phi(\sigma, q)$ is continuous on its domain. Furthermore, we have $q_0(x) \in \mathcal{Q}(x)$, since $q_0(x) \geq q_\sigma(x)$ for any $\sigma > 0$. Thus,

$$\lim_{q\to q_0(x)} \phi(\sigma, q) = \phi(\sigma, q_0(x)) \tag{A.5}$$

for any $\sigma \in [0, \bar\sigma]$. Next, note that the domain $[0, \bar\sigma] \times \mathcal{Q}(x)$ is a compact set. In addition, $\phi(\sigma, q) \leq \phi(\sigma', q)$ for all $\sigma' \leq \sigma$, since F_σ is decreasing in σ. These properties, together with the continuity of ϕ, jointly satisfy all the assumptions of Dini's theorem, and so we establish that

$$\lim_{\sigma\to 0} \phi(\sigma, q) = \phi(0, q) \tag{A.6}$$

uniformly for $q \in \mathcal{Q}(x)$. Combining (A.5)-(A.6), we may calculate the iterated limit $\lim_{q\to q_0(x)} \lim_{\sigma\to 0} \phi(\sigma, q) = \phi(0, q_0(x))$. Since the convergence in (A.6) is uniform, the Moore-Osgood theorem implies that the double limit and iterated limit coincide, i.e.,

$$\lim_{\substack{\sigma\to 0\\ q\to q_0(x)}} \phi(\sigma, q) = \lim_{q\to q_0(x)} \lim_{\sigma\to 0} \phi(\sigma, q) = \phi(0, q_0(x)). \tag{A.7}$$

Consequently, $\phi(\sigma, q_\sigma(x)) \to \phi(0, q_0(x))$, or equivalently $f_\sigma(x) \to f_0(x)$, as desired.

Next, we prove the second claim that $(f_\sigma)_{\sigma\in(0,\bar{\sigma})}$ are UI, i.e.,

$$\inf_{\alpha>0} \sup_{\sigma\in(0,\bar{\sigma})} \int_0^\eta \mathbf{1}_{\{|f_\sigma(x)|>\alpha\}} |f_\sigma(x)| dx = 0. \tag{A.8}$$

We will use the following three properties:

(P1) $f_\sigma(\cdot)$ is bounded for each $\sigma > 0$ small enough.

[Proof: see Lemma A.4 below.]

(P2) $F_\sigma(\cdot, q_0(\cdot))$ is bounded for each $\sigma > 0$ small enough.

[Proof: Recall that $q_\sigma(x)$ is decreasing in σ, and note the result from Lemma A.5 below that $\partial_q F_\sigma(x, q)\big|_{q=q_\sigma} < 0$. Together, these imply $F_\sigma(x, q_0(x)) \le F_\sigma(x, q_\sigma(x)) = f_\sigma(x)$ for any σ small enough. Then, using property (P1), we obtain the result.]

(P3) $f_0(x) = F_0(x, q_0(x))$ is finite for all $x > 0$.

[Proof: By property (P1), the associated capital share κ_σ satisfies $\kappa_\sigma(x) > x$ for all $x > 0$. Since $\kappa_\sigma(x)$ is decreasing in σ (because $q_\sigma(x)$ is decreasing in σ), we have $\kappa_0(x) \ge \kappa_\sigma(x) > x$ for all $x > 0$. This result on $\kappa_0(x)$ implies that $f_0(x) = F_0(x, q_0(x))$ is finite for all $x > 0$ (potentially unbounded at $x = 0$ since $\kappa_0(0) = 0$).]

By property (P1), we have for α large enough that

$$\sup_{\sigma\in(0,\bar{\sigma})} \int_0^\eta \mathbf{1}_{\{|f_\sigma(x)|>\alpha\}} |f_\sigma(x)| dx \le \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} |f_0(x)| dx$$

Let us now use the triangle inequality to write

$$\begin{aligned}
&\int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} |f_0(x)| dx \\
&\quad \le \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \big|F_{\tilde{\sigma}}(x, q_0(x))\big| dx + \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \big|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\big| dx
\end{aligned}$$

for any arbitrary $\tilde{\sigma} > 0$. Both of these integrals are well-defined. The first integral is well-defined by property (P2). The second integral vanishes as $\tilde{\sigma} \to 0$ by the monotone convergence theorem, since $\partial_\sigma F_\sigma(x, q) < 0$. Hence, $\left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right|$ is integrable for $\tilde{\sigma}$ small enough.

Due to property (P3), there exists a threshold $\eta^*(\alpha)$, satisfying $\lim_{\alpha\to\infty} \eta^*(\alpha) = 0$, such that $\mathbf{1}_{\{|f_0(x)|>\alpha\}} = \mathbf{1}_{\{x<\eta^*(\alpha)\}}$ for all α large enough. Thus, write for α large enough

$$\int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x))\right| dx = \int_0^{\eta^*(\alpha)} \left|F_{\tilde{\sigma}}(x, q_0(x))\right| dx$$
$$\int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right| dx = \int_0^{\eta^*(\alpha)} \left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right| dx$$

implying that

$$\begin{aligned} &\inf_{\alpha>0} \left\{ \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x))\right| dx + \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right| dx \right\} \\ &\quad = \lim_{\alpha\to\infty} \left\{ \int_0^{\eta^*(\alpha)} \left|F_{\tilde{\sigma}}(x, q_0(x))\right| dx + \int_0^{\eta^*(\alpha)} \left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right| dx \right\} = 0 \end{aligned}$$

The equality to zero is because both integrals are well-defined and $\eta^*(\alpha) \to 0$ as $\alpha \to \infty$.

Putting all these results together, we have

$$\begin{aligned} &\inf_{\alpha>0} \sup_{\sigma\in(0,\bar{\sigma})} \int_0^\eta \mathbf{1}_{\{|f_\sigma(x)|>\alpha\}} |f_\sigma(x)| dx \\ &\le \inf_{\alpha>0} \left\{ \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x))\right| dx + \int_0^\eta \mathbf{1}_{\{|f_0(x)|>\alpha\}} \left|F_{\tilde{\sigma}}(x, q_0(x)) - F_0(x, q_0(x))\right| dx \right\} = 0, \end{aligned}$$

so UI holds for $(f_\sigma)_{\sigma\in(0,\bar{\sigma})}$. Consequently, (A.4) holds, proving that q_0 solves the BSE ODE (22) on $\eta \in (0, \eta_0^*)$.

Step 3: Verify $\eta_0^* > 0$. To confirm $\eta_0^* > 0$, we use that fact that q_0 coincides with the BSE q_{BSE}. Proposition 1 has already proved that this BSE is unique and features $\eta^*_{\text{BSE}} := \inf\{\eta : q_{\text{BSE}}(\eta) = a_e/\bar{\rho}(\eta)\} > 0$, confirming our guess.

Final notes: The qualification about "convergence in distribution" is only needed because the BSE is driven by the sunspot shock Z, while the present limiting equilibrium is driven by the fundamental shock W. These shocks have the same distribution but are not pointwise identical. In addition, note that the BSE has some terms where σ is present, for instance in the expression for r in (13). When we take $\sigma \to 0$, we are also doing so in the BSE, so that all equilibrium objects in the CAE and BSE coincide as $\sigma \to 0$. □

Proof of Proposition 3. See Appendix A.3. □

A.1 Conventional Accelerator Equilibrium (CAE)

Let us collect the relevant equilibrium equations for the CAE of Definition 3. First, since the log utility consumption rules are unchanged, the price-output relation (PO) still holds. Second, solving a similar portfolio choice problem as in the text, but with the constraints $x_e = 0$ and $x_h = 0$, the expert and household capital Euler equations are now

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q - r = \frac{qk_e}{n_e}[(\sigma+\sigma_q)^2 + \varsigma_q^2]$$
$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q - r \leq \frac{qk_h}{n_h}[(\sigma+\sigma_q)^2 + \varsigma_q^2]$$

Differencing these equations leads to the risk-balance condition

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\left((\sigma+\sigma_q)^2 + \varsigma_q^2\right)\right]. \tag{A.9}$$

Summing the capital Euler equations, weighted by κ and $1-\kappa$, respectively, leads to

$$r = \bar{\rho} + g + \mu_q + \sigma\sigma_q - \left(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\right)[(\sigma+\sigma_q)^2 + \varsigma_q^2]. \tag{A.10}$$

Applying Itô's formula to the definition of η and using net worth dynamics, we have

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa-\eta}{\eta(1-\eta)}(\sigma+\sigma_q)^2 \tag{A.11}$$

$$\sigma_\eta = (\kappa-\eta)(\sigma+\sigma_q) \tag{A.12}$$

$$\varsigma_\eta = (\kappa-\eta)\varsigma_q \tag{A.13}$$

Finally, by the Itô condition, we have $\sigma_q = \frac{q'}{q}\sigma_\eta$ and $\varsigma_q = \frac{q'}{q}\varsigma_\eta$. Combining this with (A.12)-(A.13), we obtain

$$[1-(\kappa-\eta)q'/q]\sigma_q = \sigma(\kappa-\eta)q'/q \tag{A.14}$$

$$[1-(\kappa-\eta)q'/q]\varsigma_q = 0 \tag{A.15}$$

This completes the set of conditions. As the next result demonstrates, the sunspot shock must be silent in this CAE, so we may set $\varsigma_q = \varsigma_\eta = 0$ everywhere above. In that case, as mentioned in the text, the equilibrium equations for the CAE are the same as in the benchmark BSE, but with ς_q and ς_η replaced everywhere by $\sigma + \sigma_q$ and σ_η, respectively (and $\pi = 0$ used in the BSE expression for r).

Lemma A.2. *Consider a CAE with $\sigma > 0$. The sunspot shock Z must play no role: $\varsigma_\eta = \varsigma_q = 0$.*

Proof of Lemma A.2. Suppose that $\varsigma_q \neq 0$. If so, then (A.15) requires $1 = (\kappa - \eta)q'/q$. From (A.14), this then implies $\sigma_q = \pm\infty$. If σ_q is infinite, then (A.9) implies $\kappa = \eta$. Using this in the requirement $1 = (\kappa - \eta)q'/q$, we have that $q' = +\infty$. However, using $\kappa = \eta$ in (PO) yields $q' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\rho_e - \rho_h}{\bar{\rho}} q$, which contradicts $q' = +\infty$. □

Lemma A.3. *In a CAE with $\sigma > 0$, the ODE*

$$q' = F_\sigma(\eta, q) := \frac{(a_e - a_h)q}{\bar{\rho}(\eta)q - \eta a_e - (1-\eta)a_h} \left[1 - \sigma \sqrt{\frac{(\bar{\rho}(\eta)q - \eta a_e - (1-\eta)a_h)q}{\eta(1-\eta)(a_e - a_h)^2}}\right] \quad \text{(A.16)}$$

holds on $\{\eta \in (0,1) : \eta < \kappa(\eta) < 1\}$.

Proof of Lemma A.3. Combine equations (A.9) and (A.14), using the result from Lemma A.2 that $\varsigma_q = 0$, to get

$$\frac{a_e - a_h}{q} = \frac{\kappa - \eta}{\eta(1-\eta)} \left(\frac{\sigma}{1 - (\kappa - \eta)q'/q}\right)^2, \quad \text{if } \kappa < 1. \quad \text{(A.17)}$$

Next, use the CAE condition that $\sigma + \sigma_q > 0$ to take the square root of this equation and keep the positive root of the quadratic term. In addition, use equation (PO) to eliminate κ, and finally rearrange terms to get the ODE (A.16). Note that the requirement $\kappa > \eta$ is due to equation (A.9). □

A.2 Useful lemmas and uniqueness result for the CAE

Below are a few extra lemmas regarding the CAE. These serve as intermediate results helpful to the results above.

Lemma A.4. *Consider any price and capital allocation functions $(q_\sigma, \kappa_\sigma)$ from a CAE with $\sigma > 0$ and $\kappa(0) = 0$. Then, provided σ is small enough, we have that $0 < F_\sigma(\eta, q_\sigma(\eta)) < \infty$ on $\{\eta : \kappa_\sigma < 1\} \cup \{0\}$.*

Proof of Lemma A.4. To economize on notation, drop the σ subscripts and let (q,κ) denote the CAE solution we are talking about. We will make their dependence on σ clear when necessary. Our goal is to show that $0 < q' < \infty$ whenever $\kappa < 1$.

Step 1: $\kappa < 1$ for all η close enough to zero. Suppose, leading to contradiction, that $\kappa = 1$ for all η small. If $\kappa = 1$, then (PO) gives $q(\eta) = a_e/\bar{\rho}(\eta)$, implying from (A.14) that $\sigma + \sigma_q = \frac{1}{1+(1-\eta)(\rho_e-\rho_h)}\sigma > 0$. Plugging this result into (A.9) along with the guess $\kappa = 1$, we see that (A.9) is violated for all η close enough to zero. Hence, the set $\{\eta : \kappa(\eta) < 1\}$ is non-empty, and in particular $\eta^* := \inf\{\eta : \kappa(\eta) < 1\} > 0$.

Step 2: $0 < q' < \infty$ holds for all η close enough to zero. By Step 1, we have that equation (A.17) holds for all η small enough. (This equation will be simpler to work with in this step than the ODE (A.16), which is an implication.)

We first show that $|q'(0)| < \infty$. If not, then $|q'(0)| = \infty$. In that case, equation (PO) implies that $|\kappa'(0)| = \infty$. Moreover, it must be that $q'(0) = \kappa'(0) = +\infty$, for if these derivatives were $-\infty$, then $\kappa(0) = 0$ would imply that $\kappa(\eta) < 0$ for small enough η. Then, using $\kappa(0) = 0$ and applying L'Hôpital's rule to (A.17), we obtain

$$\frac{a_e - a_h}{a_h/\rho_h} = (\kappa'(0) - 1)\lim_{\eta\to 0}\Big(\frac{\sigma}{1-(\kappa-\eta)q'/q}\Big)^2 \tag{A.18}$$

For this to hold given $\kappa'(0) = +\infty$, and given the left-hand-side is finite, we must have $\lim_{\eta\to 0}(\kappa-\eta)\frac{q'}{q} = +\infty$. However, this would contradict the CAE condition that $\sigma + \sigma_q > 0$, which from equation (A.14) requires $(\kappa-\eta)\frac{q'}{q} \leq 1$.

Knowing that $|q'(0)| < \infty$ and $|\kappa'(0)| < \infty$, we may write (A.18) as

$$\kappa'(0) = 1 + \frac{\rho_h}{\sigma^2}\frac{a_e - a_h}{a_h}$$

By equation (PO), we have

$$\frac{q'(0)}{q(0)} = \frac{a_e - a_h}{a_h} - \frac{\rho_e - \rho_h}{\rho_h} + \Big(\frac{a_e - a_h}{a_h\sigma}\Big)^2\rho_h$$

which is positive if $\sigma < \sigma^\dagger$ for some $\sigma^\dagger$ small enough.

Given $0 < q'(0) < \infty$ is finite and $q'(\eta)$ is continuous for η near zero,[15] we have the existence of $\eta^\dagger > 0$ such that $0 < q'(\eta) < \infty$ for all $\eta \in [0,\eta^\dagger)$ and any $\sigma \in (0,\sigma^\dagger)$.

[15]To see this, note that the ODE (A.16) implies continuity so long as $\eta > 0$ and $\kappa > \eta$. Since $\kappa(0) = 0$ and $\kappa(0) > 1$ was derived above, we clearly have $\kappa > \eta$ for small enough η.

Step 3: $q' > 0$ holds on $\{\eta \geq \eta^\dagger : \kappa < 1\}$. Whenever ODE (A.16) holds, the condition for q being an increasing function is

$$q' > 0 \;\Leftrightarrow\; \sigma^2 q[\bar{\rho}(\eta)q - \eta a_e - (1-\eta)a_h] - \eta(1-\eta)(a_e - a_h)^2 < 0 \tag{A.19}$$

The left-hand-side of this inequality is quadratic in q, which has one positive root

$$q_+(\eta;\sigma) := \frac{1}{2\bar{\rho}(\eta)}\left[\eta a_e + (1-\eta)a_h + \sqrt{(\eta a_e + (1-\eta)a_h)^2 + 4\bar{\rho}(\eta)\eta(1-\eta)(a_e - a_h)^2\sigma^{-2}}\right]$$

Using that fact, and that $q > 0$, the condition $q' > 0$ is equivalent to

$$q < q_+(\eta;\sigma) \tag{A.20}$$

Now, for each $\eta > 0$, define $\sigma^\circ(\eta)$ by the positive solution to $\frac{a_e}{\bar{\rho}(\eta)} = q_+(\eta;\sigma)$, which after some algebra can be written

$$\sigma^\circ(\eta) = \sqrt{\frac{a_e - a_h}{a_e}\bar{\rho}(\eta)\eta}. \tag{A.21}$$

Note that $\sigma^\circ(\eta)$ is strictly positive and increasing for all $\eta > 0$. Furthermore, by the definition of $\sigma^\circ(\eta)$, and the fact that $q_+(\eta;\sigma)$ is increasing in σ, we have that $\frac{a_e}{\bar{\rho}(\eta)} < q_+(\eta;\sigma)$ for all $\sigma \in (0, \sigma^\circ(\eta))$. Because of the fact that $\kappa < 1$ when the ODE is in force, we have $q < \frac{a_e}{\bar{\rho}(\eta)}$, which establishes that (A.20) holds for all $\sigma < \sigma^\circ(\eta)$. Let $\sigma^\circ_{\min} := \inf_{\eta \geq \eta^\dagger} \sigma^\circ(\eta) = \sigma^\circ(\eta^\dagger) > 0$, by the fact that $\sigma^\circ(\eta)$ is an increasing function and $\eta^\dagger > 0$. This establishes that (A.20) holds on $\{\eta \geq \eta^\dagger\} \times \{\sigma \leq \sigma^\circ_{\min}\}$, or equivalently that $q' > 0$ on $\{\eta \geq \eta^\dagger : \kappa \in (\eta, 1)\}$, provided $\sigma \leq \sigma^\circ_{\min}$.

Step 4: $q' > 0$ holds everywhere on $\{\eta : \kappa < 1\}$. Putting Steps 2-3 together, we may now pick any $\sigma < \bar{\sigma} := \min(\sigma^\dagger, \sigma^\circ_{\min})$, so that $q' > 0$ everywhere on $\{\eta : \kappa < 1\}$.

Step 5: $q' < \infty$ holds on $\{\eta \geq \eta^\dagger : \kappa < 1\}$. This is a simple consequence of Lemma A.3, which shows that $\kappa > \eta$ whenever ODE (A.16) holds. □

Lemma A.5. *Consider any price function q_σ from a CAE with $\sigma > 0$ and $\kappa(0) = 0$. Then, provided σ is small enough, and q is close enough to $q_\sigma(\eta)$, we have that $\partial_q F_\sigma(\eta, q) < 0$.*

Proof of Lemma A.5. Note that q_σ solves the ODE (A.16). Implicitly, we are assuming q is close enough to $q_\sigma(\eta)$ in every subsequent step. First, differentiate $F_\sigma(\eta, q)$ with respect to q:

$$\frac{\partial}{\partial q}F_\sigma(\eta,q) = \Big(\frac{a_e-a_h}{\bar{\rho}(\eta)q-\bar{a}(\eta)} - \frac{(a_e-a_h)q\bar{\rho}(\eta)}{(\bar{\rho}(\eta)q-\bar{a}(\eta))^2}\Big)\Big[1-\sigma\sqrt{\frac{(\bar{\rho}(\eta)q-\bar{a}(\eta))q}{\eta(1-\eta)(a_e-a_h)^2}}\Big]$$
$$- \frac{\sigma}{2}\frac{2\bar{\rho}(\eta)q-\bar{a}(\eta)}{\eta(1-\eta)(a_e-a_h)^2}\frac{(a_e-a_h)q}{\bar{\rho}(\eta)q-\bar{a}(\eta)}\sqrt{\frac{\eta(1-\eta)(a_e-a_h)^2}{(\bar{\rho}(\eta)q-\bar{a}(\eta))q}}$$
$$< -\frac{a_e-a_h}{\bar{\rho}(\eta)q-\bar{a}(\eta)}\frac{\bar{a}(\eta)}{\bar{\rho}(\eta)q-\bar{a}(\eta)}\Big[1-\sigma\sqrt{\frac{(\bar{\rho}(\eta)q-\bar{a}(\eta))q}{\eta(1-\eta)(a_e-a_h)^2}}\Big],$$

where $\bar{a}(\eta) := \eta a_e + (1-\eta)a_h$ is defined to save space. The inequality on the third line holds because $\bar{\rho}q_\sigma - \bar{a} > 0$ (by $\kappa_\sigma > \eta$) so that the term on the second line is negative. By Lemma A.4, we have $F_\sigma(\eta, q_\sigma(\eta)) > 0$ for all σ small enough. This implies the term in square brackets is positive, and so $\partial_q F_\sigma(\eta,q) < 0$. □

Lemma A.6. *Consider a CAE with $\sigma > 0$ and $\kappa(0) = 0$. Then, provided σ is small enough, there exists a threshold $\eta^* > 0$ such that $\kappa(\eta) < 1$ for all $\eta < \eta^*$ and $\kappa(\eta) = 1$ for all $\eta \geq \eta^*$.*

Proof of Lemma A.6. Let $\sigma > 0$ be small enough. Suppose $\{\eta : \kappa < 1\}$ were not a connected set (an interval). Then, there would exist some $\eta_2 > \eta_1$ such that $q(\eta_2) < a_e/\bar{\rho}(\eta_2)$ while $q(\eta_1) = a_e/\bar{\rho}(\eta_1)$. But since $\bar{\rho}(\eta)$ is an increasing function, we have

$$q(\eta_2) < a_e/\bar{\rho}(\eta_2) < a_e/\bar{\rho}(\eta_1) = q(\eta_1)$$

This implies that $q' < 0$ for some $\eta \in (\eta_1,\eta_2) \cap \{\kappa < 1\}$, which contradicts Lemma A.4 that $q' > 0$ in the fire sale region. Hence, $\{\eta : \kappa < 1\}$ is an interval. But we also know from Step 1 of Lemma A.4 that $(0,\eta^*) \subset \{\eta : \kappa < 1\}$ for some $\eta^* > 0$. The only way these facts can both be true is that $\{\eta : \kappa < 1\} = (0,\eta^*)$. □

Lemma A.7. *For each $\sigma > 0$, at most one CAE exists satisfying $\kappa(0) = 0$.*

Proof of Lemma A.7. Given the result of Lemma A.3, it suffices to show there is at most one solution q_σ to ODE (A.16), when augmented with the boundary condition $\kappa(0) = 0$, since we can construct κ, σ_q, r, μ_η, and σ_η uniquely from q_σ, via equations (PO), (A.14), (A.10), (A.11), and (A.12), respectively.

From Lemma A.6, we have that the ODE (A.16) holds if and only if $\eta \in (0,\eta^*)$ for some endogenous threshold $\eta^* > 0$.

The rest is very similar to Step 3 of Proposition 1. Suppose two solutions q_σ and $\tilde{q}_\sigma$ solved the ODE (A.16) with boundary conditions $\kappa_\sigma(0) = \tilde{\kappa}_\sigma(0) = 0$. Let η^*_σ and $\tilde{\eta}^*_\sigma$ denote the associated fire-sale thresholds (points where $\kappa_\sigma(\eta)$ and $\tilde{\kappa}_\sigma(\eta)$ reach 1). Without loss of generality, we may consider the situation $\tilde{q}_\sigma(\eta) > q_\sigma(\eta)$ for all $\eta < \eta^\circ$. The reason: if the two solutions ever crossed at some value of η°, then they would necessarily coincide for all $\eta \geq \eta^\circ$. Since $q_\sigma(0) = \tilde{q}_\sigma(0)$, we have

$$q_\sigma(\eta) - \tilde{q}_\sigma(\eta) = \int_0^\eta \Big[F_\sigma(x, q_\sigma(x)) - F_\sigma(x, \tilde{q}_\sigma(x))\Big] dx, \quad \eta < \eta^\circ \tag{A.22}$$

By Lemma A.5, we have $\partial_q F_\sigma(\eta, q) < 0$ for all q near either q_σ or $\tilde{q}_\sigma$. Since $q_\sigma(0) = \tilde{q}_\sigma(0)$, we may take η small enough that $\partial_q F_\sigma(\eta, q) < 0$ for all q between $q_\sigma(\eta)$ and $\tilde{q}_\sigma(\eta)$. Therefore, $\tilde{q} > q$ on $(0, \eta^\circ)$ implies $F_\sigma(x, q_\sigma(x)) > F_\sigma(x, \tilde{q}_\sigma(x))$ for all x small enough, which by equation (A.22) implies $q_\sigma(\eta) > \tilde{q}_\sigma(\eta)$ for η small enough, a contradiction. □

A.3 Perceived Accelerator Equilibrium (PAE)

Let us collect the relevant equilibrium equations for the PAE of Definition 4. First, since the log utility consumption rules are unchanged, the price-output relation (PO) still holds. Second, solving a similar portfolio choice problem as in the text, but with agents perceiving capital evolution (26), the expert and household capital Euler equations are

$$\begin{aligned}
\frac{a_e}{q} + g + \tilde{\mu}_q + \sigma\tilde{\sigma}_q + \varsigma\tilde{\varsigma}_q - (\sigma + \tilde{\sigma}_q)\pi - r &= \frac{qk_e}{n_e}(\varsigma + \tilde{\varsigma}_q)^2 \\
\frac{a_h}{q} + g + \tilde{\mu}_q + \sigma\tilde{\sigma}_q + \varsigma\tilde{\varsigma}_q - (\sigma + \tilde{\sigma}_q)\pi - r &\leq \frac{qk_h}{n_h}(\varsigma + \tilde{\varsigma}_q)^2
\end{aligned}$$

In these equations $(\tilde{\mu}_q, \tilde{\sigma}_q, \tilde{\varsigma}_q)$ represent the agents' perceived price dynamics, which may differ from the actual dynamics $(\mu_q, \sigma_q, \varsigma_q)$. Differencing these equations leads to the risk-balance condition

$$0 = \min\left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)}(\varsigma + \tilde{\varsigma}_q)^2\right]. \tag{A.23}$$

Summing the capital Euler equations, weighted by κ and $1 - \kappa$, respectively, leads to

$$r = \bar{\rho} + g + \tilde{\mu}_q + \sigma\tilde{\sigma}_q + \varsigma\tilde{\varsigma}_q - (\sigma + \tilde{\sigma}_q)\pi - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)(\varsigma + \tilde{\varsigma}_q)^2 \tag{A.24}$$

Applying Itô's formula to the definition of η and using perceived net worth dynamics, we have

$$\tilde{\mu}_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}(\varsigma + \tilde{\varsigma}_q)^2 \tag{A.25}$$

$$\tilde{\sigma}_\eta = 0 \tag{A.26}$$

$$\tilde{\varsigma}_\eta = (\kappa - \eta)(\varsigma + \tilde{\varsigma}_q) \tag{A.27}$$

These differ from the actual wealth share dynamics, discussed below.

At this point, we introduce the following two concepts. Let $\tilde{\eta}$ be the perceived wealth share, and define the perceived pricing function $\tilde{q}(\cdot)$.[16] The perceived wealth share $\tilde{\eta}$ is defined implicitly by the requirement $\tilde{q}(\tilde{\eta}) = q$, since the actual capital price q is observable. We will now solve for the perceived pricing function $\tilde{q}$.

By the Itô condition, we have $\tilde{\sigma}_q = \frac{\tilde{q}'}{\tilde{q}}\tilde{\sigma}_\eta$ and $\tilde{\varsigma}_q = \frac{\tilde{q}'}{\tilde{q}}\tilde{\varsigma}_\eta$. Combining this with (A.26)-(A.27), we obtain $\tilde{\sigma}_q = 0$ and

$$[1 - (\kappa - \tilde{\eta})\tilde{q}'/\tilde{q}]\tilde{\varsigma}_q = \varsigma(\kappa - \tilde{\eta})\tilde{q}'/\tilde{q} \tag{A.28}$$

By inspection, all these equations—price-output (PO), risk-balance (A.23), riskless rate (A.24), perceived dynamics of η (A.25)-(A.27), and the perceived volatility equation (A.28)—are identical to the CAE, but with (σ, σ_q) replaced by $(\varsigma, \tilde{\varsigma}_q)$ and with the perceived pricing function $\tilde{q}$. Because of this property, any PAE (if it exists) must feature a perceived pricing function $\tilde{q}_{\text{PAE}}$ which coincides with that of the CAE, q_{CAE}, for $\varsigma = \sigma$ (which by Lemma A.7 is unique for small enough σ). Therefore, $\tilde{q}_{\text{PAE}} \to q_{\text{BSE}}$ as $\varsigma \to 0$, by identical arguments used for Proposition 2 (i.e., convergence of the pricing *function*).

Next, we consider the *actual*, as opposed to perceived, dynamics that emerge in this equilibrium for any $\varsigma > 0$. For η, use Itô's formula on the definition of η to obtain

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + \eta(1-\eta)\Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2\Big)(\varsigma + \tilde{\varsigma}_q)^2 - (\kappa - \eta)\varsigma_q^2 \tag{A.29}$$

$$\sigma_\eta = 0 \tag{A.30}$$

$$\varsigma_\eta = (\kappa - \eta)\varsigma_q \tag{A.31}$$

[16] Agents' mistaken beliefs require some measurement error in η. Indeed, if agents perceived the correct wealth share level, $\tilde{\eta} = \eta$, then by simply observing the market price of capital, they would be forced to use the correct pricing function $\tilde{q} = q$. In that case, following the subsequent algebra to its conclusion, we would find the solution q to coincide with the CAE solution, implying $\tilde{\varsigma}_q$ would coincide with the CAE volatility function. On the other hand, by equation (A.32) would imply $\varsigma_q = 0$. In that case, it would not be possible for beliefs to "converge to rational expectations", since Proposition 2 implies $\tilde{\varsigma}_q \not\to 0 = \varsigma_q$.

In the above, the perceived volatility $\varsigma + \tilde{\varsigma}_q$ appears in the drift, because this perception determines risk premia and precautionary savings, which then translates into the actual dynamics of η via its drift. Note that, as $\varsigma \to 0$, all these expressions coincide with their perceived counterparts if and only if $\tilde{\varsigma}_q \to \varsigma_q$. For q, use the Itô conditions $\sigma_q = \frac{q'}{q}\sigma_\eta$ and $\varsigma_q = \frac{q'}{q}\varsigma_\eta$ to get $\sigma_q = 0$ and

$$0 = [1 - (\kappa - \eta)q'/q]\varsigma_q \tag{A.32}$$

All these equations depend on the actual pricing function q and the actual volatility ς_q. One solution to (A.32) is the BSE pricing function q_{BSE}. Another solution is the fundamental equilibrium q_{FE}.[17] As these are independent of ς, their limiting values are also q_{BSE} and q_{FE}, respectively.

The next task is to impose convergence to rational beliefs as $\varsigma \to 0$. This requires the limiting pricing functions to coincide, i.e., $\lim_{\varsigma\to 0} |\tilde{q} - q| = 0$ in L^1. (In that case, the wealth share measurement error also vanishes, $\lim_{\varsigma\to 0} \tilde{\eta} = \eta$.) Since we know from the arguments above that $\lim_{\varsigma\to 0} \tilde{q}_{\mathrm{PAE}} = q_{\mathrm{BSE}}$, rational beliefs thus requires $\lim_{\varsigma\to 0} q_{\mathrm{PAE}} = q_{\mathrm{BSE}}$. This selects the BSE solution to (A.32) for all ς small enough. Intuitively, if the FE solution emerged in equilibrium, then perceptions as $\varsigma \to 0$ would retain a large divergence to reality.

Finally, it is easy to show that the entire PAE converges to that of the BSE (i.e., all the other objects besides q). We already know that ς_q coincides with the BSE volatility function for all ς small enough. As discussed above, the fact that $\tilde{\varsigma}_q \to \varsigma_q$ implies that $(\tilde{\mu}_\eta, \tilde{\sigma}_\eta, \tilde{\varsigma}_\eta) \to (\mu_\eta, \sigma_\eta, \varsigma_\eta)$. Lastly, (r, π, κ) converge to the BSE because the equations pinning them down are all deterministic functions of $(\varsigma, \tilde{\varsigma}_q, q, \eta)$.

[17] There may also be more solutions here. In the analysis of the BSE equations, we used the original risk-balance condition, which differs from its perceived counterpart (A.23), to obtain these as the only two solutions.

Online Appendix:
Dynamic Self-Fulfilling Fire Sales

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February 13, 2025

B Stationarity of the BSE

Lemma B.1. *In any BSE with $\rho_e > \rho_h$, the dynamics prevent η from reaching zero with probability one. Moreover, $(\eta_t)_{t\geq 0}$ has a non-degenerate stationary distribution on $(0,\eta^*]$, and when $\eta_t \in (\eta^*,1)$, it follows a deterministic path towards η^*.*

Proof of Lemma B.1. We consider the baseline model of Section 2.1 with boundary condition $\kappa(0) = \kappa_0 \in [0,1)$. As shown in Proposition 1, a BSE that is Markov in η exists uniquely given this boundary condition. For reference, we re-state the dynamics of η in such an equilibrium:

$$\mu_\eta = (\rho_e - \rho_h)\,\eta + \frac{a_e - a_h}{q}[\kappa - 2\kappa\eta + \eta^2]\mathbf{1}_{\eta<\eta^*} + (\rho_e - \rho_h)\,\eta^2 \tag{B.1}$$

$$\varsigma_\eta^2 = \eta(1-\eta)(\kappa-\eta)\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}, \tag{B.2}$$

where equation (B.2) follows from $\varsigma_\eta = (\kappa-\eta)\varsigma_q$ in (18) and $\varsigma_q^2 = \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}\mathbf{1}_{\eta<\eta^*}$ in (23). We proceed in several steps, examining dynamics of η above η^*, in a neighborhood just below η^*, and in a neighborhood just above 0.

Step 1: Dynamics for $\eta > \eta^*$. Equation (B.2) shows that $\varsigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. Thus, η it follows a deterministic path towards η^* if $\mu_\eta(\eta) < 0$ for all $\eta \in [\eta^*,1)$. Substituting $\kappa = 1$ into (B.1) and using $\rho_e > \rho_h$ delivers the result immediately. Given the deterministic transition toward η^*, we can ignore the sub-interval $(\eta^*,1)$ in our state space and instead consider only $(0,\eta^*)$.

Step 2: Setup of Feller conditions. In general, consider a one-dimensional process $(X_t)_{t\geq 0}$ with $dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dZ_t$ that is a regular diffusion on interval $(e_1,e_2) \subset \mathbb{R}$ (i.e., the dynamics of X depend only on X itself, and imply that it reaches every point in (e_1,e_2) with positive probability). Our process $(\eta_t)_{t\geq 0}$ satisfies these conditions for $e_1 = 0$ and $e_2 = \eta^*$.

In such case, we may apply Feller's boundary classification to decide whether boundaries e_1 and e_2 are inaccessible (avoided forever with probability 1) or accessible. To do so, let ϵ and x_0 be arbitrary numbers within interval (e_1, e_2). Define $s(y) := \exp(-\int_{x_0}^{y} \frac{2\mu_x(u)}{\sigma_x^2(u)} du)$ and $m(x) := \frac{2}{s(x)\sigma_x^2(x)}$. Boundary e_1 is inaccessible if and only if

$$I_1 := \int_{e_1}^{\epsilon} m(x)\Big(\int_{e_1}^{x} s(y)dy\Big)dx = +\infty.$$

Boundary e_2 is accessible if and only if

$$I_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{x}^{e_2} s(y)dy\Big)dx < +\infty.$$

We will prove these results in the next two steps.

Step 3: Dynamics near $e_2 = \eta^*$. Compute

$$\mu_\eta(\eta^*-) = -\eta^*(1-\eta^*)(\rho_e - \rho_h) + (1-\eta^*)\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}$$
$$\varsigma_\eta^2(\eta^*-) = \eta^*(1-\eta^*)^2\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}.$$

Since $\varsigma_\eta^2(\eta^*-)$ is bounded away from zero and $\mu_\eta(\eta^*-)$ is finite, it is easy to check that $I_2 < +\infty$, meaning $e_2 = \eta^*$ is an accessible boundary that is hit in finite time with positive probability. Furthermore, we may also show

$$J_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{\epsilon}^{x} s(y)dy\Big)dx < +\infty,$$

which implies $e_2 = \eta^*$ is a so-called "regular boundary" that must be included in the state space. (A regular boundary is a boundary that can be reached in finite time with positive probability.)

We must establish what occurs when η_t hits boundary point $e_2 = \eta^*$. Recall from step 1 that $\mu_\eta(\eta) < 0$ and $\varsigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. This implies that η_t can never enter the region $(\eta^*, 1)$ from η^* and that η_t will not stay at point η^* for an infinite amount of time. Consequently, the region $(0, \eta^*]$ is the ergodic set.

Step 4a: General analysis of dynamics near $e_1 = 0$. First, suppose our diffusion satisfied the following near $e_1 = 0$ (the notation $f(x) \sim g(x)$ means $\lim_{x\to 0} f(x)/g(x) = 1$):

$$\sigma_x^2(x) \sim \phi x^\beta \quad \phi > 0, \quad \beta \geq 0$$
$$\frac{\mu_x(x)}{\sigma_x^2(x)} \sim \theta x^{-\alpha}, \quad \alpha \geq 1, \quad \theta > 0.$$

As we will show below in step 4b, this asymptotic description is flexible enough to cover all cases within our model.

If $\alpha = 1$, we have, for x sufficiently small,

$$S_1(x,\theta) := \int_0^x \frac{s(y)}{s(x)} dy = \int_0^x \exp\Big[2\theta(\log(x) - \log(y))\Big] dy \qquad \text{(B.3)}$$
$$= x^{2\theta} \lim_{z\downarrow 0} \frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta},$$

so letting ϵ be sufficiently small, we obtain

$$I_1 = \int_0^\epsilon \frac{2x^{2\theta-\beta}}{\phi} \lim_{z\downarrow 0} \frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta} dx.$$

If $2\theta \geq 1$ (note that $2\theta = 1$ corresponds to $\frac{z^{1-2\theta}}{1-2\theta}$ being replaced by $\log(z)$ in the expression above), then the interior limit is $+\infty$ for all $x > 0$ and therefore $I_1 = +\infty$. This holds independently of the value of β. If $2\theta < 1$, then

$$I_1 = \int_0^\epsilon \frac{2}{(1-2\theta)\phi} x^{1-\beta} dx = \frac{2}{(1-2\theta)\phi}\Big(\frac{\epsilon^{2-\beta}}{2-\beta} - \lim_{x\downarrow 0} \frac{x^{2-\beta}}{2-\beta}\Big).$$

So, in this case, $I_1 = +\infty$ only if $\beta \geq 2$ (for $\beta = 2$, $\frac{x^{2-\beta}}{2-\beta}$ is replaced by $\log(x)$).

If $\alpha > 1$ instead, we will show that $I_1 = +\infty$ independent of any other parameters. We have

$$S_\alpha(x,\theta) := \int_0^x \frac{s(y)}{s(x)} dy = \int_0^x \exp\Big[\frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})\Big] dy \qquad \text{(B.4)}$$

The corresponding expression for the case with $\alpha = 1$ is $S_1(x,\theta)$ in (B.3). We showed above that for $\tau > 1/2$, we have $S_1(x,\tau) = +\infty$. Fix such a τ. We now show that $S_\alpha(x,\theta) \geq S_1(x,\tau)$ for all x sufficiently small and all θ.

Fix any $x > 0$, and define $f(y) := 2\tau(\log(x) - \log(y))$ and $g(y) := \frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})$. Since both functions are strictly positive for $y < x$, and since $\lim_{y\to 0} g(y)/f(y) = \lim_{y\to 0}(\theta/\tau)y^{1-\alpha} = +\infty$, there exists $\bar{y} \in (0,x)$ such that $g(y) > f(y)$ for all $y \in (0,\bar{y})$. From this comparison, we conclude $S_\alpha(\bar{y},\theta) = \int_0^{\bar{y}} \exp(g(y))dy \geq \int_0^{\bar{y}} \exp(f(y))dy = S_1(\bar{y},\tau) = +\infty$. Since this argument is independent of (β,θ,ϕ), this proves that $I_1 = +\infty$ if $\alpha > 1$.

Step 4b: Model-specific analysis of dynamics near $e_1 = 0$. Now, we map our model dynamics into the setup of step 4a. If $\kappa(0) = \kappa_0 > 0$, then in the limit as $\eta \to 0$, equations (B.1)-(B.2) become

$$\mu_\eta = \frac{a_e - a_h}{q(0)}\kappa_0 - \left(\rho_e - \rho_h + 2\frac{a_e - a_h}{q(0)}\kappa_0\right)\eta + o(\eta)$$
$$\sigma_\eta^2 = \frac{a_e - a_h}{q(0)}\kappa_0\eta + o(\eta).$$

Hence, in terms of the notation in step 4a, we have $\alpha = 1$, $\beta = 1$ and $\theta = 1 > \frac{1}{2}$. Thus, η avoids zero with probability one.

If $\kappa(0) = 0$, we need to know the rate at which $\kappa \to 0$ as $\eta \to 0$. Guess, and verify after, that $\kappa = \varphi\eta^\omega + o(\eta^\omega)$ in the limit as $\eta \to 0$. Differentiating the price-output condition (PO), we have

$$q' = \frac{1}{\bar{\rho}}\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q\right]$$

Combining this with the sunspot differential equation for q, equation (21), we obtain

$$\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q\right](\kappa - \eta) = \bar{\rho}q.$$

Taking the limit as $\eta \to 0$, we have

$$(a_e - a_h)\lim_{\eta\to 0}(\kappa')(\kappa - \eta) = a_h$$

Hence, the guess is verified if $\omega = 1/2$ and $\varphi^2 = 2a_h/(a_e - a_h) > 0$. Substituting this asymptotic behavior into equations (B.1)-(B.2), we have

$$\mu_\eta = \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h\eta^{1/2} + o(\eta^{1/2})$$
$$\sigma_\eta^2 = \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h\eta^{3/2} + o(\eta^{3/2}).$$

These dynamics match step 4a with $\alpha = 1$, $\beta = 3/2$, and $\theta = 1$. In that case, we have shown that η cannot reach zero with probability one.

In summary, $(\eta_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution with support $(0, \eta^*]$, the boundary $\{0\}$ is inaccessible, and the boundary η^* is accessible but non-absorbing. $\square$

C Multiplicity in the conventional financial accelerator

This section investigates properties of conventional equilibria where sunspot shocks dZ are irrelevant but fundamental shocks σdW are non-hedgeable (with $\sigma > 0$). This is the setting studied in Brunnermeier and Sannikov (2014), and its equations are contained in Appendix A.1. We illustrate novel multiplicity along two dimensions: the sign of the sensitivity of capital returns to fundamental shocks $\sigma + \sigma_q$, and agents' belief about the worst-case scenario $\kappa_0 := \kappa(0)$.

The indeterminacy in the sign of $\sigma + \sigma_q$ relates to the conjecture in footnote 16 in Kiyotaki and Moore (1997): one type of equilibrium is the "normal equilibrium" studied by the literature in which negative shocks reduce asset prices, while the second type of equilibrium is a "hedging equilibrium" in which, due to coordinated capital purchases/sales, asset prices and output respond oppositely to shocks. The presence of this indeterminacy is why the definition of Conventional Accelerator Equilibria (CAE), i.e., Definition 3, restricts attention to $\sigma + \sigma_q > 0$.

The second indeterminacy, regarding the disaster belief κ_0, is similar in spirit to the disaster belief indeterminacy documented in Section 2.3 for the BSEs. As a result of this indeterminacy, the convergence results in Propositions 2-3 made the assumption that $\kappa_0 = 0$.

We conclude this section with a simple refinement, based on a vanishingly-small limited commitment friction, that selects $\kappa_0 = 0$ as the only possible disaster belief that could be associated with an equilibrium. This refinement result implies that $\kappa_0 = 0$ is natural assumption.

Because we will make repeated references to them, let us restate the two key equations of the conventional equilibrium, which are equations (PO) in the main text and equations (A.9) and (A.14) in Appendix A.1 (with $\varsigma_q = 0$ due to Lemma A.2). These are

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{C.1}$$

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\right] \tag{C.2}$$

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma \tag{C.3}$$

This is system of three equations in the three objects (q, κ, σ_q), and so it can be solved in some sense separately from the equations characterizing $(r, \mu_\eta, \sigma_\eta)$.

C.1 The "hedging" equilibrium

Conventional Accelerator Equilibria are "normal" in the sense that a positive fundamental shock increases asset prices and experts' wealth share. But technically, agents do not care about the direction prices move when they make their portfolio choices. They only care about risk which is measured in return variance; this can be seen in the optimality condition (A.9) where $(\sigma + \sigma_q)^2$ appears. This suggests that two types of equilibria are possible: the "normal" one has $\sigma + \sigma_q > 0$; an alternative equilibrium has $\sigma + \sigma_q < 0$. Because $\sigma > 0$, this means the alternative equilibrium must counter-intuitively have $\sigma_q < 0$ We term this latter equilibrium the "hedging" equilibrium because asset price movements move oppositely to exogenous shocks.

Mathematically, we need only solve a slightly different capital price ODE. Whereas ODE (A.16) holds in the normal equilibrium, the hedging equilibrium requires

$$q' = \frac{(a_e - a_h)q}{\bar{\rho}q - \eta a_e - (1-\eta)a_h}\left[1 + \sigma\sqrt{\frac{(\bar{\rho}q - \eta a_e - (1-\eta)a_h)q}{\eta(1-\eta)(a_e - a_h)^2}}\right], \tag{C.4}$$

on $\{\eta : \eta < \kappa(\eta) < 1\}$. The difference between (C.4) and (A.16) is merely the sign in front of σ, which ensures different signs for σ_q in the region when $\kappa < 1$. While we don't provide an existence proof, Figure C.1 displays a numerical example of a hedging equilibrium and compares it to a normal equilibrium. Notice that $\sigma_q < 0$ as claimed.

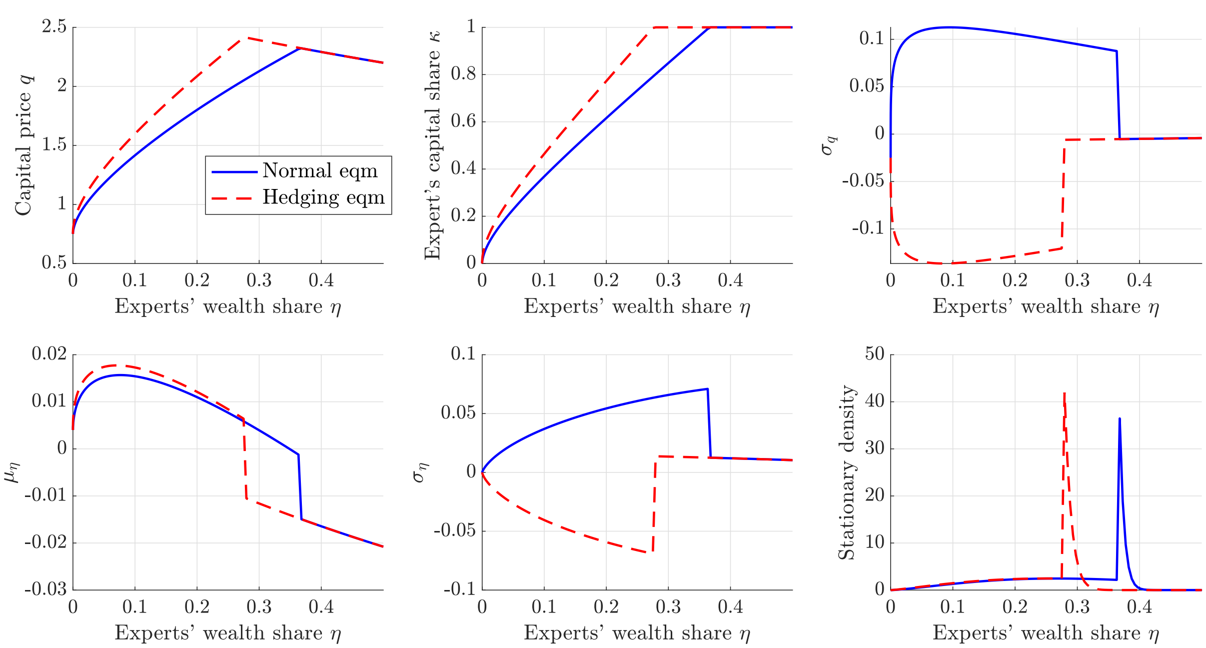

Figure C.1: Two equilibria (normal versus hedging) both with boundary condition $\kappa(0) = 0$. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

C.2 "Disaster belief" indeterminacy

The existing literature always imposes $\kappa(0) := \lim_{\eta\to 0}\kappa(\eta) = 0$, i.e., experts fully deleverage as their wealth vanishes. We have already shown that this boundary condition is indeterminate in BSEs. Here, we show the same is true for conventional equilibria.

Let $\kappa_0 \in [0,1)$ and suppose $\kappa(0) = \kappa_0$. As in Section 2.3 for BSEs, we will call κ_0 the *disaster belief* about experts' deleveraging. Existence of an equilibrium with such disaster belief boils down simply to existence of a solution to a first-order ODE with a given boundary condition $\kappa(0) = \kappa_0$.

Lemma C.1. *Let $\sigma > 0$, and suppose fundamental shocks are non-hedgeable. An equilibrium with disaster belief $\kappa_0 \in [0,1)$ exists if the free boundary problem*

$$q' = \frac{(a_e - a_h)q}{\bar{\rho}q - \eta a_e - (1-\eta)a_h}\Big[1 - \sigma\sqrt{\frac{(\bar{\rho}q - \eta a_e - (1-\eta)a_h)q}{\eta(1-\eta)(a_e-a_h)^2}}\Big], \quad \text{on} \quad \eta \in (0,\eta^*), \tag{C.5}$$

$$\text{subject to} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} \quad \text{and} \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \tag{C.6}$$

has a solution. In this equilibrium, $(\eta_t)_{t\geq 0}$ is strictly positive with probability 1.

Proof of Lemma C.1. An equilibrium in state variable η exists if and only if equations (C.1)-(C.3) hold, and if the time-paths $(\eta_t)_{t\geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (C.5)-(C.6) has a solution (q, η^*) corresponding to $\kappa_0 \in [0,1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0,\eta^*)$, which is always possible because the boundary conditions (C.6) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (C.1) is automatically satisfied. Note that (C.3) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (C.2) holds separately on $(0,\eta^*)$ and $[\eta^*,1)$. For $\eta < \eta^*$, we just plug (C.1) and (C.3) into the ODE (C.5) and rearrange, which shows that the second term of (C.2) equals zero. Since $\eta < \eta^*$ corresponds to $\kappa < 1$, this proves that (C.2) holds on $(0,\eta^*)$.

On the set $[\eta^*,1)$, we have $\kappa = 1$, so (C.2) requires

$$\eta\frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all} \quad \eta \geq \eta^*. \tag{C.7}$$

First, we show that it suffices to verify this condition exactly at η^*. Indeed, on $(\eta^*,1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (C.3) into (C.7), we obtain

$$\text{(C.7) holds} \;\Leftrightarrow\; \Big(\frac{a_e - a_h}{a_e\sigma^2}\rho_e - \frac{\rho_e - \rho_h}{\rho_e}\Big)\eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all} \quad \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η, if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (C.7) at η^*, using (C.3) to replace $\sigma + \sigma_q(\eta^*+) = \frac{\sigma}{1-(1-\eta^*)q'(\eta^*+)/q(\eta^*)}$, and using ODE (C.5) to replace $\eta^*\frac{a_e - a_h}{q(\eta^*)} = \frac{\sigma}{1-(1-\eta^*)q'(\eta^*-)/q(\eta^*)}$, we need to verify

$$\text{(C.7) holds} \;\Leftrightarrow\; \frac{\sigma}{1-(1-\eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1-(1-\eta)q'(\eta^*+)/q(\eta^*)} \;\Leftrightarrow\; q'(\eta^*-) \geq q'(\eta^*+).$$

But we clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to very that η_t almost-surely never reaches the boundary 0. Near $\eta = 0$, the dynamics in (A.11)-(A.12) become

$$\mu_\eta(\eta) = \kappa_0\frac{a_e - a_h}{q(0)} + o(\eta)$$
$$\sigma_\eta^2(\eta) = \kappa_0\frac{a_e - a_h}{q(0)}\eta + o(\eta).$$

By the same analysis as in Lemma B.1, the boundary 0 is unattainable. □

What happens in an equilibrium of Lemma C.1 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (C.2) can only hold there if $\sigma_q \to -\sigma$ as $\eta \to 0$. Capital prices "hedge" fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (C.3), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.[18]

[18]One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$. Now, assume there is an $\hat{\eta} \in (0,1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (C.5) it must be that $q'(\hat{\eta}-) \to \infty$. But this implies that q crosses $\tilde{q}$ from below, contradicting $q(\eta) > \tilde{q}(\eta)$ on $\eta \in (0, \hat{\eta})$.

Figure C.2 displays several numerical examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The other curves, with higher disaster beliefs κ_0, are new to the literature. More optimistic disaster beliefs raise capital prices and reduce capital price volatility. In fact, although there is exogenous fundamental risk, agents can coordinate on κ_0 sufficiently high, such that the equilibrium behaves arbitrarily closely to an efficient equilibrium with κ almost always equal to 1.

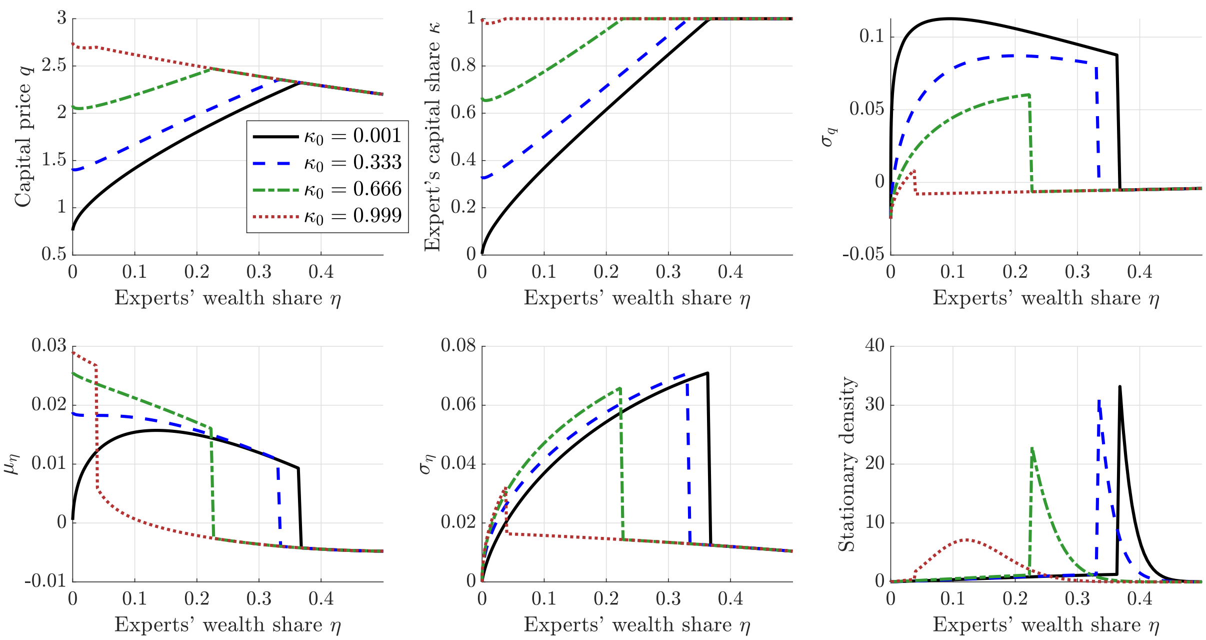

Figure C.2: Conventional equilibria with different disaster beliefs κ_0. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

C.3 Limited commitment as a refinement

Here, we add a limited commitment friction, in the spirit of Gertler and Kiyotaki (2010). This extension will serve as a refinement that prunes all possible disaster beliefs κ_0, with the exception of $\kappa_0 = 0$.

Suppose capital holders can abscond with a fraction $\beta^{-1} \in (0, 1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, the capital holder would have net worth $\tilde{n}_{j,t} := \beta^{-1} q_t k_{j,t}$. To prevent diversion, bondholders will impose some limitation on borrowing. To see this, note that diversion delivers utility $\log(\tilde{n}_{j,t}) + \xi_t$, where ξ_t is an aggregate process (independent of the identity j of the diverter). This is the form of indirect utility for a log utility investor in our model, as shown at the beginning of Appendix A. For diversion to be sub-optimal, it must be the case that $\log(\tilde{n}_{j,t}) + \xi_t \leq \log(n_{j,t}) + \xi_t$. As a result, bondholders impose the following leverage constraint to ensure non-diversion is incentive compatible:

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \beta. \tag{C.8}$$

We will study the equilibrium with constraint (C.8) additionally imposed, and then we will take $\beta \to \infty$ so that the limited commitment friction is vanishingly small.

Risk-balance condition (C.2) is now replaced by

$$0 = \min\left[1-\kappa,\ \beta\eta-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}(\sigma+\sigma_q)^2\right]. \tag{C.9}$$

The most important feature of equation (C.9) is that leverage constrained experts ($\beta\eta = \kappa$) must hold less than the full capital stock ($\kappa < 1$).

Condition (C.9) implies that there exists a threshold $\eta^{\beta}_{\kappa_0} := \inf\{\eta : \beta\eta > \kappa\}$, possibly dependent on the disaster belief κ_0, below which experts' leverage constraints bind. By combining $\beta\eta = \kappa$ with equation (C.1) for κ, we obtain an explicit formula for the capital price in this region:

$$q = q_\beta := \frac{\beta\eta a_e + (1-\beta\eta)a_h}{\bar{\rho}}, \quad \text{if} \quad \eta \leq \eta^{\beta}_{\kappa_0}. \tag{C.10}$$

In addition, the expression for q_β in (C.10) serves as an upper bound for the capital price, because $q > q_\beta$ would violate the leverage constraint.

Leverage constraint β and disaster belief κ_0, we define a *Leverage-constrained Accelerator Equilibrium* LAE(κ_0, β) as a Markov equilibrium satisfying the conditions above and $\kappa(0) = \kappa_0$. For comparison, we denote by CAE(κ_0) the Conventional Accelerator Equilibrium with disaster belief κ_0 (as in Figure C.2).

We construct a candidate LAE(κ_0, β) as follows. Let $q_{\mathrm{CAE}(\kappa_0)}$ be the capital price from the CAE(κ_0). By the leverage constraint (C.8), or equivalently $q \leq q_\beta$, we consider a candidate capital price

$$\hat{q}_{\beta,\kappa_0} := \min[q_\beta, q_{\mathrm{CAE}(\kappa_0)}]. \tag{C.11}$$

So long as $q_{\mathrm{CAE}(\kappa_0)}$ is unique, (C.11) is the unique pricing function which could satisfy the additional requirements imposed by the leverage constraint. Furthermore, $\hat{q}_{\beta,\kappa_0}$ automatically satisfies all the relevant equations, with the exception of (C.9): it remains to verify that $\frac{a_e - a_h}{q} \geq \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2$ when the leverage constraint binds (i.e., this condition says that experts would like to buy more capital than their leverage constraint allows). In particular, after substituting $\sigma + \sigma_q$ from (C.3) and $q = q_\beta$, it remains to verify that

$$\frac{a_e - a_h}{q_\beta}\Big[1 - (\beta - 1)\eta \frac{q'_\beta}{q_\beta}\Big]^2 \geq \frac{(\beta - 1)\sigma^2}{1 - \eta} \quad \text{on} \quad \{\eta : q_\beta < q_{\mathrm{CAE}(\kappa_0)}\}. \tag{C.12}$$

If (C.12) holds, then $\hat{q}_{\beta,\kappa_0}$ is the LAE(κ_0, β) pricing function. Otherwise, there cannot be any LAE(κ_0, β).

The non-existence of an LAE(κ_0, β) is precisely how the equilibrium refinement will work here. We specifically show that, for any $\kappa_0 > 0$, there exists a β large enough so that (C.12) fails. By contrast, if $\kappa_0 = 0$, then (C.12) holds for all β large enough.

Before delving into the formal proof of these claims, we provide an illustration in Figure C.3 to understand the idea. The figure shows that $q_{\mathrm{CAE}(\kappa_0)} > q_\beta$ for all $\eta < \eta^\beta_{\kappa_0}$; in other words, the CAE would violate the leverage constraint for all low enough values of η. However, there is another threshold $\hat{\eta}^\beta$ as well, defined as the smallest η such that (C.12) is violated:

$$\hat{\eta}^\beta := \inf\Big\{\eta \geq 0 : \frac{a_e - a_h}{q_\beta}\Big[1 - (\beta - 1)\eta \frac{q'_\beta}{q_\beta}\Big]^2 < \frac{(\beta - 1)\sigma^2}{1 - \eta}\Big\}. \tag{C.13}$$

If $\hat{\eta}^\beta < \eta^\beta_{\kappa_0}$, then there cannot be an equilibrium, because the candidate pricing function $\hat{q}_{\beta,\kappa_0}$ violates (C.12) for at least some values of η in the region $(\hat{\eta}^\beta, \eta^\beta_{\kappa_0})$. In Figure C.3, this describes the situation. A valid equilibrium, instead, requires that $\hat{\eta}^\beta \geq \eta^\beta_{\kappa_0}$.

The task for the proof is to take the limit as $\beta \to \infty$, so that the limited-commitment problem vanishes. A priori, it is not obvious that this helps refine equilibria. Indeed, the leverage constraint becomes non-binding at all times (formally $\eta^\beta_{\kappa_0} \to 0$).[19] This means that the condition (C.12) needs to be checked on a vanishing set of η values, suggesting non-existence issues could vanish for any κ_0. It turns out, however, that $\hat{\eta}^\beta \to 0$ faster than $\eta^\beta_{\kappa_0} \to 0$, so that non-existence persists for every κ_0 except $\kappa_0 = 0$.

Proposition C.1. *If β is sufficiently large, the unique Leverage-constrained Accelerator Equilibrium is LAE($0, \beta$). Thus, as $\beta \to \infty$, the unique equilibrium converges to CAE(0).*

[19] This intuitive property can be shown easily by taking $\beta \to \infty$ in (C.10). For any fixed $\eta \in (0, 1)$, taking this limit implies $q \to \infty$, which is ruled out by price-output relation (C.1).

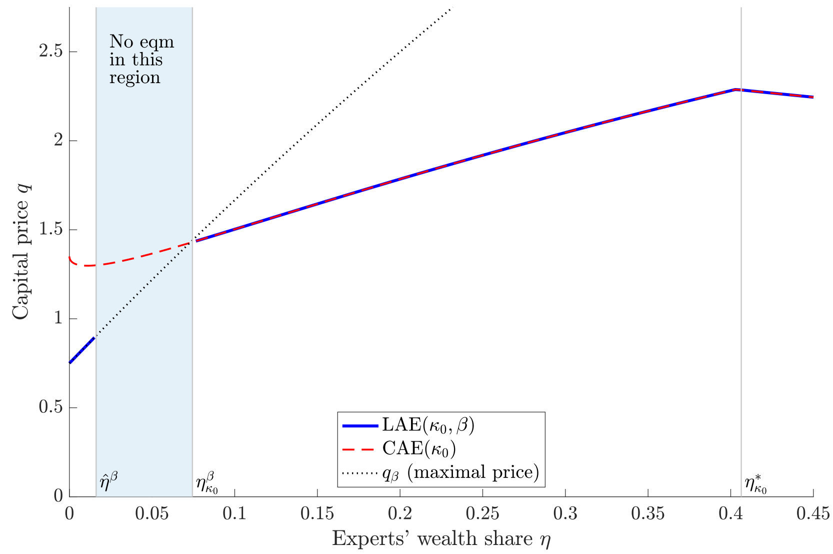

Figure C.3: A comparison of a candidate Leverage-constrained Accelerator Equilibrium LAE(κ_0, β) against a Conventional Accelerator Equilibrium CAE(κ_0). Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.05$, $\beta = 5$. The disaster belief is set to $\kappa_0 = 0.3$.

Proof of Proposition C.1. For reference, differentiate q_β and record the result:

$$\frac{q'_\beta(\eta)}{q_\beta(\eta)} = -\frac{\rho_e - \rho_h}{\bar{\rho}(\eta)} + \frac{\beta(a_e - a_h)}{a_h + \beta\eta(a_e - a_h)}$$

First, fix $\kappa_0 > 0$. We will show that $\hat{\eta}^\beta < \eta^\beta_{\kappa_0}$ for all β large enough. Indeed, $\eta^\beta_{\kappa_0} > 0$ for any $\beta < \infty$. This follows from the fact that $q_\beta(0) = \frac{a_h}{\rho_h} < \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} = q_{\text{CAE}(\kappa_0)}(0)$ and that $q'_\beta(\eta)$ is finite for all $\beta < \infty$ and all $\eta \in [0,1]$. Hence, $\eta^\beta_{\kappa_0} = \inf\{\eta : q_\beta > q_{\text{CAE}(\kappa_0)}\} > 0$. On the other hand, $\hat{\eta}^\beta = 0$ for all β large enough. To see this, use again that $q'_\beta(\eta)$ is finite, and then take $\eta \to 0$ in (C.12) to get $1 - (\beta - 1)\eta q'_\beta / q_\beta \to 1$, so that

$$\lim_{\eta \to 0} \frac{a_e - a_h}{q_\beta}\left[1 - (\beta - 1)\eta\frac{q'_\beta}{q_\beta}\right]^2 < \lim_{\eta \to 0} \frac{(\beta - 1)\sigma^2}{1 - \eta} \iff \beta > 1 + \rho_h \frac{a_e - a_h}{a_h \sigma^2}.$$

This proves that $\kappa_0 > 0$ cannot be consistent with an equilibrium with leverage constraint β, for all β large enough.

Next, we prove that $\kappa_0 = 0$ remains an equilibrium for all β large enough. Indeed, refer to Step 2 of the proof of Lemma A.4, which shows that for $\kappa_0 = 0$,

$$\frac{q'_{\text{CAE}(0)}(0)}{q_{\text{CAE}(0)}(0)} = \frac{a_e - a_h}{a_h} - \frac{\rho_e - \rho_h}{\rho_h} + \left(\frac{a_e - a_h}{a_h \sigma}\right)^2 \rho_h,$$

which is finite. By contrast, $q'_\beta(\eta) \to +\infty$ as $\beta \to +\infty$, for all $\eta \in [0,1]$. Since $q_\beta(0) = q_{\text{CAE}(0)}(0) = a_h/\rho_h$, we have that for all β large enough, it holds that $q_\beta(\eta) > q_{\text{CAE}(0)}(\eta)$ for all $\eta > 0$. Thus, $\eta_0^\beta = \inf\{\eta : q_\beta > q_{\text{CAE}(0)}\} = 0$ for all β large enough. This is thus a valid equilibrium, which completes the proof. $\square$

Remark C.1. *Notice that this limited commitment refinement does not work without fundamental volatility (i.e., $\sigma = 0$). In other words, the same argument cannot be applied directly to the BSEs to select $\kappa_0 = 0$. Indeed, when $\sigma = 0$, inequality (C.12) holds trivially for every β, implying that experts are happy to be leverage constrained and that there is no contradiction to equilibrium. Intuitively, agents may coordinate on $\sigma_q = 0$ when the leverage constraint binds, precisely because $\sigma = 0$, and therefore capital is effectively risk-free in that region.*

D General CRRA preferences

We modify the model by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in utility specification (3) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. We impose no fundamental volatility, $\sigma = 0$, to simplify the expressions.

Equilibrium. The key equation (21) still holds, repeated here for convenience, but in terms of ς_η rather than ς_q:

$$\left[1 - (\kappa - \eta)\frac{q'}{q}\right]\varsigma_\eta = 0. \tag{D.1}$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q,\eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta)K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma - 1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i. Then, goods market clearing becomes

$$q^{1/\gamma}\left[\left(\frac{\eta}{v_e}\right)^{1/\gamma} + \left(\frac{1-\eta}{v_h}\right)^{1/\gamma}\right] = (a_e - a_h)\kappa + a_h. \tag{D.2}$$

Optimal portfolio decisions imply that

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \left(\frac{v_h'}{v_h} - \frac{v_e'}{v_e} + \frac{1}{\eta(1-\eta)}\right)(\kappa-\eta)\varsigma_q^2\right]. \tag{D.3}$$

The HJB equation for $i \in \{e, h\}$ has the familiar form $\rho_i V_i = u(c) + \mathbb{E}[\frac{dV_i}{dt}]$, which becomes

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\mu_\eta + \frac{1}{2}\frac{v_i''}{v_i}\varsigma_\eta^2 + (1-\gamma)g. \tag{D.4}$$

The dynamics of η satisfy

$$\varsigma_\eta = (\kappa - \eta)\varsigma_q \tag{D.5}$$

$$\mu_\eta = \eta(1-\eta)\left(\pi_e \frac{\kappa}{\eta}\varsigma_q - \pi_h \frac{1-\kappa}{1-\eta}\varsigma_q + \frac{c_h}{n_h} - \frac{c_e}{n_e}\right) - \varsigma_\eta \varsigma_q \tag{D.6}$$

and agent-specific risk prices satisfy

$$\pi_e = -\frac{v_e'}{v_e}\varsigma_\eta + \frac{\varsigma_\eta}{\eta} + \varsigma_q \tag{D.7}$$

$$\pi_h = -\frac{v_h'}{v_h}\varsigma_\eta - \frac{\varsigma_\eta}{1-\eta} + \varsigma_q. \tag{D.8}$$

A Markov equilibrium is a set of prices $\{q, \sigma_q, \pi_e, \pi_h\}$, allocation $\{\kappa\}$, value functions $\{v_h, v_e\}$ and aggregate state dynamics $\{\varsigma_\eta, \mu_\eta\}$ that solve the system (D.1)-(D.8).

The Fundamental Equilibrium corresponds to the solution for (D.1) where $\varsigma_\eta = 0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\varsigma_q = 0$), risk prices are zero ($\pi_e = \pi_h = 0$), and experts hold the entire capital stock ($\kappa = 1$). The capital price is then solved from (D.2), and the value functions satisfy

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\underbrace{\eta(1-\eta)\left(\frac{c_h}{n_h} - \frac{c_e}{n_e}\right)}_{=\mu_\eta} + (1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (D.1) with $\frac{q'}{q} = (\kappa - \eta)^{-1}$ (and potentially $\varsigma_\eta \neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\varsigma_q(0) = 0$. This allowed us, in Section 2.3, to construct sunspot equilibria with $\kappa(0) = \kappa_0$ for any $\kappa_0 \in [0, 1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\varsigma_q(0) = 0$ and $\kappa_0 \in [0, 1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\varsigma_\eta(0) = \kappa_0 \varsigma_q(0) = 0$, the state variable avoids the boundary $\{0\}$ if $\mu_\eta(0) > 0$. Using (D.3) for $\kappa < 1$, we have[20]

$$\frac{a_e - a_h}{q(0)} = (\pi_e(0) - \pi_h(0))\varsigma_q(0)$$

which allows us to show that[21]

$$\mu_\eta(0) = \kappa_0 \frac{a_e - a_h}{q(0)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.

Numerical solution. We do not provide an existence proof but construct numerical examples. For numerical stability, the examples are constructed for $\kappa_0 > 0$, which keeps $q'(0) = q(0)/\kappa_0$ bounded.[22]

The numerical strategy is the following. Construct a grid $\{\eta_1, \ldots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (D.2) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (D.2) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover ς_q from (D.3). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\varsigma_q = 0$, and solve capital price from (D.2). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (D.4) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure D.1, we plot the equilibrium objects as functions of η, for different levels of risk aversion γ. In Figure D.2, we make the same plots, for different levels of the disaster belief κ_0. Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0) generates sunspot equilibria featuring lower capital prices and higher volatility.

[20] Note that this implies $\pi_e(0) - \pi_h(0)$ diverges.

[21] This expression also assumes that $\pi_h(0)$ remains bounded. This is a mild assumption that is always confirmed numerically when we solve for the value functions.

[22] With logarithmic utility, we obtain a limiting result in Proposition 1, that as $\kappa_0 \to 0$, the equilibrium converges to the BSE with $\kappa(0) = 0$. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

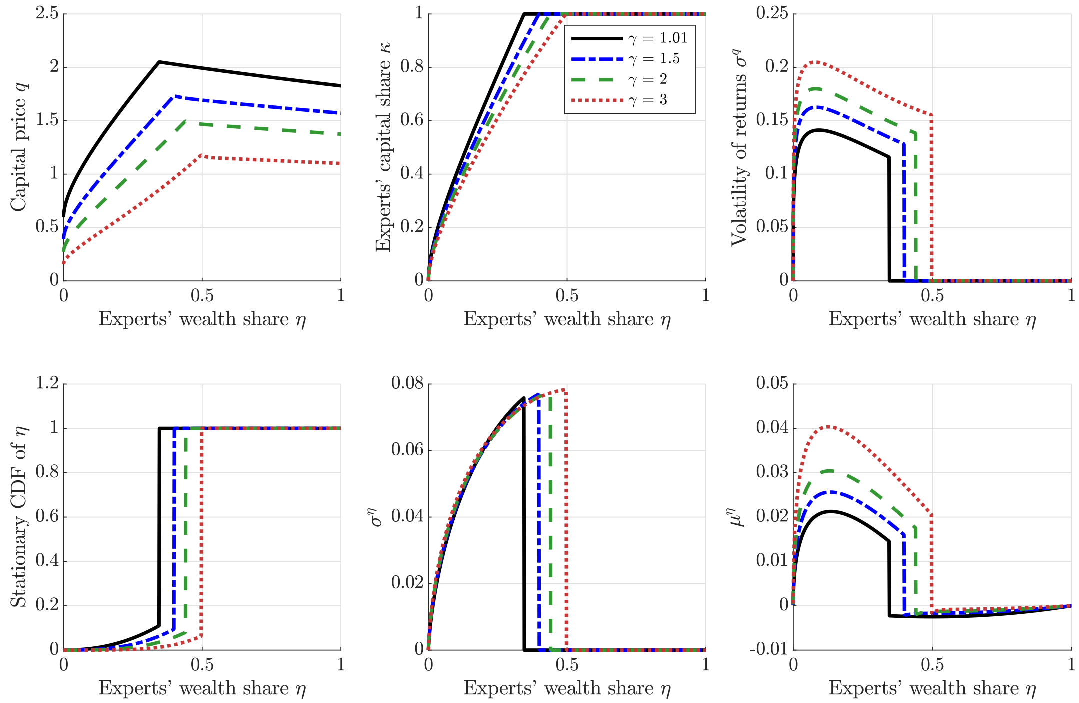

Figure D.1: Sunspot equilibrium for different risk aversion γ. The disaster belief is set to $\kappa_0 = 0.001$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

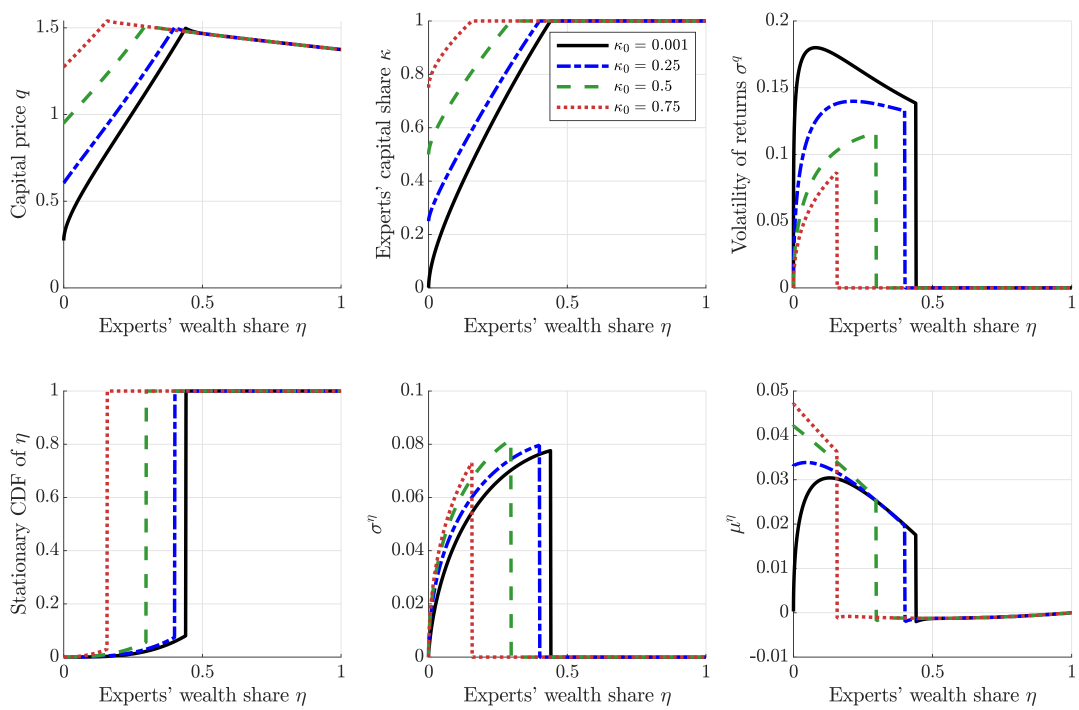

Figure D.2: Sunspot equilibrium for different disaster beliefs κ_0. Risk aversion is set to $\gamma = 2$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

E More details on the Poisson Sunspot Equilibrium (PSE)

This appendix contains some additional details and derivations for the Poisson jump setup of Section 4.

Optimal portfolios. The derivation of the portfolio choice formulas is straightforward but tedious. Heuristically, proceed as follows. Note that the return-on-capital for an agent with productivity a is given by

$$dR = \frac{a}{q} + \frac{d(qk)}{qk} = (\frac{a}{q} + g + \mu_q)dt + \frac{\hat{q}\hat{k} - qk}{qk}dJ$$
$$= \underbrace{(\frac{a}{q} + g + \mu_q)}_{:=\mu_R} dt - (\zeta + \zeta_q - \zeta\zeta_q)dJ,$$

where variables with hats, e.g., "$\hat{x}$", denote post-jump levels. The net worth evolution, and net worth jump, of any agent is

$$dn = (nr - c)dt + qk(dR - rdt)$$
$$\hat{n}/n = 1 - \frac{qk}{n}(\zeta + \zeta_q - \zeta\zeta_q)$$

Using these, we may derive the following HJB equation for an agent's value function V, which is a function of individual net worth n and the aggregate wealth distribution η:

$$\rho V = \max_{c,k\geq 0} \log(c) + (nr - c + qk\mu_R)\partial_n V + \mu_\eta \partial_\eta V + \lambda(\hat{V} - V)$$

(This step is dramatically simplified by the fact that post-jump values like $\hat{V}$ are known ex-ante.) We can guess and verify that $V(n, \eta) = \rho^{-1}\log(n) + \xi(\eta)$ for some function ξ which is type-specific but independent of individual wealth. Plug this guess into the HJB equation, take the FOC with respect to consumption to obtain the familiar rule $c = \rho n$, and plug everything back in to obtain:

$$\rho\xi = \max_{k\geq 0} \log(\rho) + \rho^{-1}(r - \rho) + \rho^{-1}\frac{qk}{n}(\mu_R - r) + \lambda\rho^{-1}\log\left(1 - \frac{qk}{n}(\zeta + \zeta_q - \zeta\zeta_q)\right)$$
$$+ \mu_\eta\partial_\eta\xi + \lambda(\hat{\xi} - \xi)$$

From here, we take the FOC with respect to qk/n, subject to $k \geq 0$, to obtain the Euler equations in the text. Then, we can either plug qk/n back into the HJB, or we can simply note that all experts/households choose the same and substitute $qk_e/n_e = \kappa/\eta$ and $qk_h/n_h = (1-\kappa)/(1-\eta)$. As a final point, note that we need to verify our guess for the value function, which amounts to verifying that a pair of functions ξ_e, ξ_h (one for experts, one for households) exists satisfying the coupled equation system above. We assume the existence of such solutions without proof.

Some other equilibrium objects. The full set of endogenous objects in the jump model is $(q, \kappa, \zeta_R, \zeta_q, \zeta_\eta, r, \mu_\eta)$, which is 7 functions of η (given a Markov equilibrium). In the text, equations (27)-(28)-(29) and (RBJ) are 4 of equations needed. We can complete the set of equations for the other equilibrium objects similarly to before. The goods market clearing condition yields the same price-output relation (PO) as before, since optimal consumption rules are still $c = \rho n$,

$$\bar{\rho} q = \kappa a_e + (1-\kappa) a_h.$$

The riskless rate is given by taking a $(\kappa, 1-\kappa)$-weighted sum of the two agents' Euler equations, to get

$$r = \frac{\kappa a_e + (1-\kappa) a_h}{q} + g + \mu_q - \lambda \zeta_R \Big(\frac{\kappa}{1 - \frac{\kappa}{\eta}\zeta_R} + \frac{1-\kappa}{1 - \frac{1-\kappa}{1-\eta}\zeta_R} \Big).$$

Finally, by applying Itô's formula with jumps, the drift of η is obtained as

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + \frac{(\kappa - \eta)\lambda \zeta_R}{\big(1 - \frac{\kappa}{\eta}\zeta_R\big)\big(1 - \frac{1-\kappa}{1-\eta}\zeta_R\big)}$$

Combining these 3 equations with the 4 equations in Section 4, we have 7 equilibrium equations for the 7 unknown objects. Assuming that q is solely a function of η as usual, this is sufficient to solve for an equilibrium. While we do not prove any existence/uniqueness results here, we provide a numerical algorithm below.

Numerical method. The numerical method for obtaining an equilibrium solution is analogous to the differential equation approach in the Brownian model, but is more cumbersome. The description of the algorithm is as follows. We solve for the function $q(\eta)$ on a grid $\eta \in \{0, \eta_1, \eta_2, \ldots, \eta_N, 1\}$. The procedure starts from the boundary condition $\kappa(0) = 0$ to obtain $q_0 = q(0) = a_h/\rho_h$. Then, we perform a numerical search for $q_1 = q(\eta_1)$, requiring that the resulting post-jump values of the wealth share and capital price $(\hat{\eta}_1, \hat{q}_1)$ lie on the interpolated function $\{0, \eta_1\} \mapsto \{q_0, q_1\}$, i.e., $\hat{q}_1 \approx q(\hat{\eta}_1)$. Once this is approximately true, within some tolerance, we move on to the next grid-point η_2 and numerically search for $q_2 = q(\eta_2)$ such that the post-jump values $(\hat{\eta}_2, \hat{q}_2)$ lie on the interpolated function $\{0, \eta_1, \eta_2\} \mapsto \{q_0, q_1, q_2\}$. And so on.

The post-jump values $(\hat{\eta}_n, \hat{q}_n)$ in each of these steps are determined as follows. First, the risk-balance equation (RBJ) is a quadratic equation in ζ_R with two roots; the requirement that $\zeta_R \in (0,1)$ guides the choice of which root to pick.[23] Then, given ζ_R, equation (27) pins down $\zeta_q = \frac{\zeta_R - \zeta}{1-\zeta}$, hence the post-jump asset price $\hat{q}_n$. Similarly, given ζ_R, equation (28) pins down ζ_η, hence the post-jump wealth share $\hat{\eta}_n$. The procedure terminates once we reach the "efficient region" when capital is no longer misallocated: once we reach a grid-point n^* such that $q_{n^*} > a_e/\bar{\rho}(\eta_{n^*})$, we then set $\kappa(\eta_n) = 1$ and $q(\eta_n) = a_e/\bar{\rho}(\eta_n)$ for all $n \geq n^*$.

[23] After rearranging the second term in the minimum of (RBJ), we obtain the following quadratic equation

$$0 = A\zeta_R^2 - \frac{\kappa(1-\eta) + \eta(1-\kappa)}{\eta(1-\eta)}\zeta_R + 1$$
$$\text{where} \quad A := \frac{1}{\eta(1-\eta)}\Big(\kappa(1-\kappa) - \frac{\lambda q(\kappa - \eta)}{a_e - a_h}\Big)$$

This quadratic equation has two roots: ζ_R^+ and ζ_R^-. It is easy to verify that $\zeta_R^+ > 0$ always. Indeed, if the economy is such that $A > 0$, then both roots are positive. If the economy is such that $A = 0$, then the unique root is positive. If the economy is such that $A < 0$, then the roots have opposite signs. Therefore, we pick the larger root for $\zeta_R = \zeta_R^+$, unless it yields $\zeta_R^+ \geq 1$, in which case we select the smaller root, if it is positive. If no root is lies in $(0,1)$, then we update q to a new value.

F Partial equity-issuance

We extend the model to allow some equity issuance by capital holders, subject to a constraint. Let us consider the model with fundamental risk $\sigma > 0$ and no direct hedging markets, in which case Lemma A.2 implies that the sunspot shock can play no role. As before, we focus on the conventional case $\sigma + \sigma_q > 0$. Hence, the model we analyze here is akin to the original Brunnermeier and Sannikov (2014) model but with partial equity issuance, as analyzed in Brunnermeier and Sannikov (2016).

In particular, at any point of time, agents managing capital can issue some equity to the market, but the issuer must keep at least $\chi \in [0, 1]$ fraction of their capital risk—this is a so-called "skin-in-the-game" constraint. In other words, if experts and households retain χ_e and χ_h of their capital risk, respectively, it must be the case that

$$\chi_{i,t} \geq \chi, \quad i \in \{e, h\}. \tag{F.1}$$

Thus, the frictionless model corresponds to $\chi = 0$, while our baseline model corresponds to $\chi = 1$. Outside equity contracts are risky, having risk exposure $\sigma + \sigma_q$ (the endogenous capital return volatility), so they must promise an excess return $(\sigma + \sigma_q)\pi$, where π is the equilibrium risk price vector.

Agents' dynamic budget constraints are now given by

$$\begin{aligned} dn_{i,t} = & \Big[(n_{i,t} - q_t k_{i,t})r_t - c_{i,t} + a_i k_{i,t}\Big]dt + d(q_t k_{i,t}) \\ & + [x_{i,t} - (1 - \chi_{i,t})q_t k_{i,t}](\sigma + \sigma_{q,t})(\pi_t dt + dW_t). \end{aligned} \tag{F.2}$$

The second line of (F.2) contains the new terms pertaining to equity-issuance: $x_{i,t} \geq 0$ denotes purchases of equity contracts in the market, while $\chi_{i,t}$ denotes the fraction of capital risk retained. Notice that it will be without loss of generality to assume $\chi_{i,t} = \chi$ at all times and for all agents, because the purchase variable $x_{i,t}$ is available as a control. For example, an agent with a slack equity-issuance constraint ($\chi_i > \chi$) could issue equity to the constraint (F.1) and then buy back such exposure by increasing their x_i control. Going forward, we thus put $\chi_{e,t} = \chi_{h,t} = \chi$. The presence of a public equity market implies an additional market clearing condition for equity securities, namely

$$x_{e,t} + x_{h,t} = (1 - \chi)q_t K_t. \tag{F.3}$$

At this point, we may solve for equilibrium.

Equilibrium characterization. The introduction of equity issuance changes nothing about optimal consumption choices, so the price-output relation (PO) still holds.

Optimal portfolio choice now implies the following four FOCs:

$$\mu_{R,e} - (1-\chi)(\sigma+\sigma_q)\pi - r = \chi\Big(\frac{\chi q k_e}{n_e} + \frac{x_e}{n_e}\Big)(\sigma+\sigma_q)^2 \tag{F.4}$$

$$\mu_{R,h} - (1-\chi)(\sigma+\sigma_q)\pi - r \leq \chi\Big(\frac{\chi q k_h}{n_h} + \frac{x_h}{n_h}\Big)(\sigma+\sigma_q)^2, \quad \text{with equality if } k_h > 0 \tag{F.5}$$

$$\Big(\frac{\chi q k_e}{n_e} + \frac{x_e}{n_e}\Big)(\sigma+\sigma_q) \geq \pi, \quad \text{with equality if } x_e > 0 \tag{F.6}$$

$$\Big(\frac{\chi q k_h}{n_h} + \frac{x_h}{n_h}\Big)(\sigma+\sigma_q) \geq \pi, \quad \text{with equality if } x_h > 0 \tag{F.7}$$

where $\mu_{R,i} := \frac{a_i}{q} + g + \mu_q + \sigma\sigma_q$ is the expected return on capital for agent i. Equations (F.4)-(F.5) are the FOCs for capital holdings, and (F.6)-(F.7) are the FOCs for equity purchases. Note that the equality in (F.4) assumes $k_e > 0$, which is must always be the case in equilibrium because experts obtain a higher output from holding capital.

By analyzing these conditions, we are able to derive the following characterization.

Lemma F.1. *The economy with constrained equity-issuance features two regions.*

(i) If $\eta > \chi$, then capital allocation and risk-sharing are efficient, with $\kappa = 1$, $\sigma_\eta = 0$, $\sigma_q = 0$, and $\pi = \sigma$. The economy exits this region deterministically in finite time, since $\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) < 0$.

(ii) If $\eta \leq \chi$, then risk-sharing is incomplete, and efficient capital allocation may fail. The equilibrium objects (q, κ, σ_q) jointly satisfy equations (PO) and

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \chi\frac{\chi\kappa - \eta}{\eta(1-\eta)}(\sigma+\sigma_q)^2\Big] \tag{F.8}$$

$$\sigma_q = \frac{(\chi\kappa - \eta)q'/q}{1-(\chi\kappa-\eta)q'/q}\sigma \tag{F.9}$$

The dynamics of η are stochastic in this region, with $\sigma_\eta \neq 0$.

Proof of Lemma F.1. There will essentially be three regions to consider: (i) $\kappa < 1$ (capital misallocation); (ii) $\kappa = 1$ and $x_e = 0$ (efficient capital allocation but constrained risk-sharing); and (iii) $\kappa = 1$ and $x_e > 0$ (efficient capital allocation and risk-sharing).

First, suppose $\kappa < 1$ (so that both $k_e > 0$ and $k_h > 0$). If experts are selling capital, then they must be constrained in their hedging activities, and so $x_e = 0$.[24] Using $x_e = 0$

[24] If $x_e > 0$, then equations (F.4)-(F.7) would imply $\mu_{R,e} = \mu_{R,h}$, in contradiction to $\mu_{R,e} > \mu_{R,h}$.

in the equity market clearing condition (F.3) implies $\frac{x_h}{n_h} = \frac{1-\chi}{1-\eta}$. Substituting these results into (F.4)-(F.5) and differencing these equations, we obtain

$$\frac{a_e - a_h}{q} = \chi \frac{\chi\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2, \quad \text{if } \kappa < 1. \tag{F.10}$$

Next, by (F.7) and the derived expression $\frac{x_h}{n_h} = \frac{1-\chi}{1-\eta}$, we have

$$\pi = \frac{1-\chi\kappa}{1-\eta}(\sigma + \sigma_q), \quad \text{if } \kappa < 1. \tag{F.11}$$

Using this expression for π, (F.6) requires $\chi\kappa \geq \eta$ when $\kappa < 1$, which holds automatically by equation (F.10). In particular, $\kappa < 1$ can only arise $\eta < \chi$ (i.e., $\kappa = 1$ if $\eta \geq \chi$).

Now, suppose $\kappa = 1$, which has the two sub-cases $x_e = 0$ and $x_e > 0$. These sub-cases correspond to $\eta < \chi$ and $\eta \geq \chi$, respectively. To show this, consider the two cases:

(i) If $x_e = 0$, then market clearing (F.3) implies $\frac{x_h}{n_h} = \frac{1-\chi}{1-\eta} > 0$. Using this expression in (F.7), we have $\pi = \frac{1-\chi}{1-\eta}(\sigma + \sigma_q)$. Using this π and $\kappa = 1$, we find that (F.6) holds if and only if $\eta \leq \chi$.

(ii) If $x_e > 0$, then (F.6)-(F.7), $\kappa = 1$, and market clearing (F.3) imply $\frac{x_e}{n_e} = 1 - \chi/\eta$ and $\frac{x_h}{n_h} = 1$.[25] This expression for x_e is positive, as required, if and only if $\eta > \chi$. Plugging x_h back in (F.7), we obtain $\pi = \sigma + \sigma_q$.

Intuitively, experts are only constrained in their hedging activities if when they issue maximal equity, their risk share is greater than their wealth share. In the process of these derivations, we also obtained the risk price

$$\pi = \min\left(1, \frac{1-\chi}{1-\eta}\right)(\sigma + \sigma_q), \quad \text{if } \kappa = 1. \tag{F.12}$$

And we may use the results just obtained in the capital FOCs (F.4)-(F.5), and then differencing them as before, to obtain

$$\frac{a_e - a_h}{q} \geq \chi \frac{\chi\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2, \quad \text{if } \kappa = 1. \tag{F.13}$$

Combining (F.10) and (F.13) leads to a new "risk-balance" condition (F.8), analogously to the baseline model. One implication of (F.8) is that $\kappa < 1$ must arise for all η is low

[25] Note that $x_h > 0$ must hold. Indeed, $x_e > 0$ implies $\pi > 0$ via (F.6) while $k_h = x_h = 0$ implies the opposite via (F.7).

enough (provided $\sigma + \sigma_q > 0$, which we will verify below). Therefore, as claimed in the lemma, $\eta < \chi$ bears the possibility of $\kappa < 1$.

Next, putting the results of (F.11)-(F.12) together, we have that

$$\pi = \begin{cases} \sigma + \sigma_q, & \text{if } \eta > \chi; \\ \frac{1-\chi\kappa}{1-\eta}(\sigma + \sigma_q), & \text{if } \eta \leq \chi. \end{cases} \tag{F.14}$$

The riskless interest rate can be derived as always, by summing a $(\kappa, 1-\kappa)$-weighted-average of equations (F.4)-(F.5) and using the results above to get[26]

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q - (\sigma+\sigma_q)^2 - \Big(\frac{\chi\kappa}{\eta} - 1\Big) \max\Big(0, \frac{\chi\kappa - \eta}{1-\eta}\Big). \tag{F.15}$$

The dynamics of the wealth share η are derived via applying Itô's formula to its definition, and using all the previous results. After significant algebra, this yields

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\chi\kappa - 2\eta\chi\kappa + \eta^2)\frac{\chi\kappa - \eta}{\eta(1-\eta)}(\sigma+\sigma_q)^2 \mathbf{1}_{\{\eta \leq \chi\}} \tag{F.16}$$

$$\sigma_\eta = (\chi\kappa - \eta)(\sigma + \sigma_q)\mathbf{1}_{\{\eta \leq \chi\}} \tag{F.17}$$

Finally, if we assume q is a function of η, then we obtain the Itô condition $\sigma_q = \frac{q'}{q}\sigma_\eta$, which combined with (F.17) implies equation (F.9). This completes the derivation of equilibrium, and the displayed equations can be used to verify all the remaining claims of the lemma. $\square$

Observationally near-equivalent asset prices. We can provide a precise sense in which the equilibrium with equity-issuance looks approximately like the conventional accelerator equilibrium (CAE) for any $\chi > 0$. To do this, it is helpful to consider a change-of-variables allowing a proper comparison.

The dynamics of η in (F.16)-(F.17) reveal that the ergodic set for η_t is $(0, \chi]$. Indeed, recall that $\mu_\eta < 0$ and $\sigma_\eta = 0$ for all $\eta \geq \chi$. So for an economy with equity-retention χ,

[26] To derive (F.15), start by summing a $(\kappa, 1-\kappa)$-weighted-average of equations (F.4)-(F.5) to get

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q - (1-\chi)(\sigma+\sigma_q)\pi - \chi\Big[\kappa\Big(\frac{\chi\kappa}{\eta} + \frac{x_e}{n_e}\Big) + (1-\kappa)\Big(\frac{\chi(1-\kappa)}{1-\eta} + \frac{x_h}{n_h}\Big)\Big](\sigma+\sigma_q)^2.$$

We can simplify this equation using the following facts. First, from the discussion above, $x_h > 0$ always holds, so that (F.7) holds with equality, hence $\frac{x_h}{n_h} = \frac{\pi}{\sigma+\sigma_q} - \frac{\chi(1-\kappa)}{1-\eta}$. Next, we may use the market clearing condition (F.3) to obtain $\frac{x_e}{n_e} = \frac{1-\chi}{\eta} - \frac{1-\eta}{\eta}\frac{x_h}{n_h}$. We use these two facts to eliminate x_e and x_h from the equation above, then we substitute the solution for π from (F.14), and finally we simplify the result to obtain (F.15).

the appropriate region to consider is $(0, \chi]$. Define the transformed state variable

$$\omega := \frac{\eta}{\chi}, \tag{F.18}$$

whose ergodic distribution has the fixed support $[0, 1]$ for any χ. By ignoring values of $\omega > 1$, which have probability zero in the ergodic distribution, we have a state space whose domain is invariant to χ. Also define $\hat{q}(\omega) = q(\chi\omega) = q(\eta)$, the pricing function under the change-of-variables. Note that $\hat{q}'(\omega) = \chi q'(\eta)$. Similarly define $\hat{\kappa}(\omega) = \kappa(\chi\omega)$.

Using these change-of-variables, and combining equations (F.8)-(F.9), we obtain the ODE that holds on $\{\omega : \hat{\kappa} < 1\}$:

$$\frac{a_e - a_h}{\hat{q}} = \frac{\hat{\kappa} - \omega}{\chi\omega(1 - \chi\omega)} \Big(\frac{\chi\sigma}{1 - (\hat{\kappa} - \omega)\hat{q}'/\hat{q}} \Big)^2 \tag{F.19}$$

Define $\Sigma(\omega; \chi, \sigma) := \sqrt{\frac{\chi - \chi\omega}{1 - \chi\omega}}\sigma$. Using this definition in (F.19), we have

$$\frac{a_e - a_h}{\hat{q}} = \frac{\hat{\kappa} - \omega}{\omega(1 - \omega)} \Big(\frac{\Sigma(\omega; \chi, \sigma)}{1 - (\hat{\kappa} - \omega)\hat{q}'/\hat{q}} \Big)^2 \tag{F.20}$$

Inspecting this equation, we see that it is very similar to the CAE differential equation (A.17) in Lemma A.3, but with two differences. First, the exogenous volatility σ replaced by the endogenous volatility $\Sigma(\omega; \chi, \sigma)$. And second, there is an adjustment to the wealth-weighted average discount rate: one must replace $\bar{\rho}(\eta)$ with $\bar{\rho}(\chi\omega)$, since from the goods market clearing condition we have

$$\hat{\kappa}(\omega) = \frac{\hat{q}(\omega)\bar{\rho}(\chi\omega) - a_h}{a_e - a_h}. \tag{F.21}$$

Since the two mathematical differences relative to the CAE are the volatility σ and the discount rate $\bar{\rho}$, we may control these to provide the following bounds on the equilibrium pricing function.

Proposition F.1. *Suppose that, for each χ, functions $\hat{q}_\chi : [0, 1] \mapsto \mathbb{R}$ and $\hat{\kappa}_\chi : [0, 1] \mapsto [0, 1]$ exist satisfying (F.19), (F.21), and $\hat{\kappa}_\chi(0) = 0$. Let $q_{CAE}(x; \sigma, \rho_e)$ denote the CAE price at experts wealth share is x, and with exogenous volatility σ and expert discount rate ρ_e. Then, for σ small enough,*

$$q_{CAE}(\omega; \sigma, \rho_e) \leq \hat{q}_\chi(\omega; \sigma, \rho_e) \leq q_{CAE}(\omega; \Sigma(\omega^*_\chi; \chi, \sigma), \rho_h), \quad \text{for all } \ \omega \in [0, 1],$$

where $\omega^*_\chi := \inf\{\omega : \hat{\kappa}_\chi(\omega) = 1\}$ is the fire-sale threshold.

Proof of Proposition F.2. We fix σ, ρ_e, and χ, and let $\hat{q}_\chi(\omega;\sigma,\rho_e)$ denote a solution to (F.19), (F.21), and $\hat{\kappa}_\chi(0)=0$. Define $\omega^*_\chi := \inf\{\omega : \hat{\kappa}_\chi(\omega)=1\}$.

Consider the CAE ODE from Lemma A.3, but with ω as the independent variable, $\hat{\sigma}$ as the exogenous volatility (to differentiate from the parameter σ), and $\hat{\rho}_e$ as the expert discount rate (to differentiate from the parameter ρ_e), and denoting $\hat{q}$ as the unknown function:

$$\begin{aligned}\frac{1}{\hat{q}} &= \frac{\hat{q}[\omega\hat{\rho}_e+(1-\omega)\rho_h]-\omega a_e-(1-\omega)a_h}{\omega(1-\omega)} \\ &\quad\times\Big(\frac{\hat{\sigma}}{a_e-a_h-(\hat{q}[\omega\hat{\rho}_e+(1-\omega)\rho_h]-\omega a_e-(1-\omega)a_h)\hat{q}'/\hat{q}}\Big)^2. \end{aligned} \tag{F.22}$$

First, consider the two CAE solutions. The function $q_{\text{CAE}}(\omega;\sigma,\rho_e)$ is a solution to (F.22) with $\hat{\sigma}=\sigma$ and $\hat{\rho}_e=\rho_e$. The function $q_{\text{CAE}}(\omega;\Sigma(\omega^*_\chi;\chi,\sigma),\rho_h)$ is a solution to (F.22) with $\hat{\sigma}=\Sigma(\omega^*_\chi;\chi,\sigma)$ and $\hat{\rho}_e=\rho_h$.

On the other hand, the function $\hat{q}_\chi$ also solves a version of (F.22). Notice that its ODE (F.20) is identical to (F.22) but with the volatility function $\Sigma(\omega;\chi,\sigma)$ replacing the constant $\hat{\sigma}$ and with the wealth-weighted average discount rate $\bar{\rho}(\chi\omega)=\omega(\chi\rho_e+(1-\chi)\rho_h)+(1-\omega)\rho_h$ replacing $\bar{\rho}(\omega)$. This establishes that $\hat{q}_\chi(\omega;\sigma,\rho_e)$ is a solution to (F.22) with $\hat{\sigma}=\Sigma(\omega;\chi,\sigma)$ and $\hat{\rho}_e=\chi\rho_e+(1-\chi)\rho_h$.

Now, we establish the bounds using monotonicity properties of (F.22), since all three functions in question solve a version of that ODE. First, note the monotonic structure of (F.22): both $\hat{\sigma}$ and $\hat{\rho}_e$ affect $\hat{q}'$ negatively.

The elementary inequalities $\chi\rho_e+(1-\chi)\rho_h \le \rho_e$ and $\Sigma(\omega;\chi,\sigma)\le\sigma$ hold for all ω. Together with the monotonicity properties above, these inequalities imply the lower bound $q_{\text{CAE}}(\omega;\sigma,\rho_e)\le\hat{q}_\chi(\omega;\sigma,\rho_e)$ for all $\omega\in[0,1]$.

On the other hand, using the inequalities $\chi\rho_e+(1-\chi)\rho_h\ge\rho_h$ and $\Sigma(\omega;\chi,\sigma)\ge\Sigma(\omega^*_\chi;\chi,\sigma)$, the latter of which holds for all $\omega\le\omega^*_\chi$, along with the same monotonicity properties of the ODE (F.22), we have that $\hat{q}_\chi(\omega;\sigma,\rho_e)\le q_{\text{CAE}}(\omega;\Sigma(\omega^*_\chi;\chi,\sigma),\rho_h)$ for $\omega\le\omega^*_\chi$. We can extend this inequality to all $\omega\in[0,1]$ using the assumption that σ is small enough. Indeed, by Lemmas A.4, A.6, and A.7, $q_{\text{CAE}}(\omega;\Sigma(\omega^*_\chi;\chi,\sigma),\rho_h)$ is the unique CAE solution with those parameters, and it is necessarily monotonically increasing until it hits its unique fire-sale threshold ω^*_{CAE}.

Putting these inequalities together, we obtain the result. □

In words, Proposition F.1 shows that the pricing function from the economy with equity-retention χ is (on its ergodic set) between two CAE solutions without any equity-issuance: one with the same parameters but evaluated with the transformed state variable $\omega = \eta/\chi$, and another with a lower level of risk and a lower expert discount rate. Therefore, if σ and $\rho_e - \rho_h$ are small, the solution with equity-issuance must approximately coincide with a CAE (at least in the sense that the equilibrium pricing functions are close). Furthermore, due to Proposition 2, which shows that the CAE converges to the BSE as $\sigma \to 0$, we thus have that an equilibrium with any amount of equity-retention $\chi > 0$ necessarily converges to the BSE as $\sigma \to 0$.

Near-perfect financial markets. Now, we use Lemma F.1 and Proposition F.1 to investigate what happens when $\chi \to 0$, i.e., when financial markets converge to their frictionless level. On the one hand, because $\kappa = 1$ for all $\eta \geq \chi$, we have that $\lim_{\chi \to 0} \kappa(\eta) = 1$ for all $\eta > 0$. In this sense, the equilibrium converges to the safe Fundamental Equilibrium (FE) as financial markets improve. On the other hand, the ergodic set for an economy with equity-issuance is $\{\eta \in (0, \chi]\}$. By taking $\chi \to 0$, we are shrinking this region, and in the limit almost all values of η become irrelevant to the ergodic set. This makes it unclear whether or not, in terms of the long-run distribution, fluctuations persist in the small-χ economy.

As before, settling the question of what happens when $\chi \to 0$ is only possible when using the transformed state variable $\omega = \eta/\chi$. This provides a convenient normalization, in the sense that the ergodic set is $\{\omega \in [0,1]\}$ under the change-of-variables.

But these differences are relatively minor for near-perfect markets: as $\chi \to 0$, notice that $\Sigma \to 0$ uniformly (since $\Sigma(\omega;\chi) \leq \sqrt{\chi}\sigma$) and $\bar{\rho}(\chi\omega) \to \rho_h$ uniformly. Therefore, a reasonable conjecture is that the limiting solution to (F.20) as $\chi \to 0$ will coincide with the limiting CAE as $\sigma \to 0$ if discount rates are symmetric ($\rho_e = \rho_h$). By Proposition 2, the latter is simply the BSE with symmetric discount rates. Hence, one expects that $\lim_{\chi \to 0} \hat{q}(\omega) = q_{\text{BSE}}(\omega; \rho_e = \rho_h)$. (Notice that we are using the BSE function with the variable ω as the input.) By this argument, a non-trivial fire-sale region should survive in the limit $\chi \to 0$. Filling in some technical details, we are able to prove

Proposition F.2. *Let σ be small enough. Suppose that, for each χ small enough, functions $\hat{q}_\chi : [0,1] \mapsto \mathbb{R}$ and $\hat{\kappa}_\chi : [0,1] \mapsto [0,1]$ exist satisfying (F.19), (F.21), and $\hat{\kappa}_\chi(0) = 0$. As $\chi \to 0$, the fire-sale region of the ergodic set does not vanish, in the sense that*

$$\lim_{\chi \to 0} \omega^*_\chi > 0.$$

where $\omega^*_\chi := \inf\{\omega : \kappa_\chi(\omega) = 1\}$.

Proof of Proposition F.2. As assumed, suppose there is a solution $\hat{q}_\chi$ for each χ small enough. Let $\omega^*_\chi := \inf\{\omega : \kappa_\chi(\omega) = 1\}$. By a similar argument as the CAE Lemma A.6, it must be the case that $\omega_\chi^* > 0$.

Using Proposition F.1, we have that

$$\hat{q}_\chi(\omega; \sigma, \rho_e) \le q_{\text{CAE}}(\omega; \Sigma(\omega_\chi^*; \chi, \sigma), \rho_h), \tag{F.23}$$

where $q_{\text{CAE}}(\omega; \hat{\sigma}, \hat{\rho}_e)$ represents the CAE solution with expert wealth share ω, exogenous volatility $\hat{\sigma}$, and expert discount rate $\hat{\rho}_e$. Note that $\lim_{\chi \to 0} \Sigma(\omega_\chi^*; \chi, \sigma) = 0$. Since χ only affects $q_{\text{CAE}}(\omega; \Sigma(\omega_\chi^*; \chi, \sigma), \rho_h)$ through $\Sigma(\omega_\chi^*; \chi, \sigma)$, we know by Proposition 2 that

$$\lim_{\chi \to 0} q_{\text{CAE}}(\omega; \Sigma(\omega_\chi^*; \chi, \sigma), \rho_h) = \lim_{\sigma \to 0} q_{\text{CAE}}(\omega; \sigma, \rho_h) = q_{\text{BSE}}(\omega; \rho_h), \tag{F.24}$$

where $q_{\text{BSE}}(\omega; \hat{\rho}_e)$ is the BSE solution from Proposition 1 with expert wealth share ω and expert discount rate $\hat{\rho}_e$. As established by Proposition 1, we know that $q_{\text{BSE}}(\omega; \rho_h) < a_e/\rho_h$ for all ω small enough (i.e., the fire-sale region is non-trivial). If $\rho_e - \rho_h$ is small enough, then we also have $q_{\text{BSE}}(\omega; \rho_h) < a_e/\rho_e$ for all ω small enough.[27]

Taking the limit $\chi \to 0$ in (F.23), using the result (F.24), and using the result $q_{\text{BSE}}(\omega; \rho_h) < a_e/\rho_e$ for all ω small enough, we have

$$\lim_{\chi \to 0} \hat{q}_\chi(\omega; \sigma, \rho_e) < \frac{a_e}{\rho_e}, \quad \text{for all } \omega \text{ small enough.}$$

Using equation (F.21) for the associated capital share $\hat{\kappa}_\chi$, this proves that $\hat{\kappa}_\chi < 1$ for all ω small enough. Thus, $\lim_{\chi \to 0} \omega_\chi^* > 0$. □

Remark F.1. *While Proposition F.2 is informative about the limit $\chi \to 0$, we are unable to analytically obtain the long-run amount of time spent in fire-sale, because volatilities explode.*

Indeed, equation (F.19) implies that $\lim_{\chi \to 0} (\hat{\kappa}_\chi - \omega)\hat{q}'_\chi/\hat{q}_\chi = 1$ for all $\omega \in (0, \omega_0^*)$, i.e., in the fire-sale region. Using the change-of-variables $\eta = \chi\omega$ in (F.9), we have the price volatility

$$\sigma_q = \frac{(\hat{\kappa} - \omega)\hat{q}'/\hat{q}}{1 - (\hat{\kappa} - \omega)\hat{q}'/\hat{q}} \sigma \tag{F.25}$$

But if $\lim_{\chi \to 0} (\hat{\kappa}_\chi - \omega)\hat{q}'_\chi/\hat{q}_\chi = 1$ in the fire-sale region, then $\sigma_q(\omega) \to +\infty$ in this region.

[27] For a direct proof of this, one can also appeal to also the closed-form solution to the BSE for this symmetric discount rate case $\rho_e = \rho_h = \rho$, given by

$$q(\eta) = \frac{1}{\rho}\left[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2\kappa_0^2}\right], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}(1 - \kappa_0^2).$$

As κ_0 decreases, the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster.

This explosive volatility translates into analytically-challenging state dynamics. Using the change-of-variables $\eta = \chi\omega$ in (F.16)-(F.17), and substituting (F.25), we have the following drift and diffusion of ω:

$$\mu_\omega = \omega(1-\chi\omega)(\rho_h - \rho_e) + (\hat{\kappa} - 2\chi\omega\hat{\kappa} + \chi\omega^2)\frac{\hat{\kappa}-\omega}{\omega(1-\chi\omega)}\Big(\frac{\sigma}{1-(\hat{\kappa}-\omega)\hat{q}'/\hat{q}}\Big)^2 \quad \text{(F.26)}$$

$$\sigma_\omega = \frac{(\hat{\kappa}-\omega)\sigma}{1-(\hat{\kappa}-\omega)\hat{q}'/\hat{q}} \quad \text{(F.27)}$$

Since $(\hat{\kappa}_\chi - \omega)\hat{q}'_\chi/\hat{q}_\chi \to 0$ in the fire-sale region, both σ_ω and μ_ω explode there. A more careful analysis must examine the rates of explosion for the drift and diffusion to understand whether or not the state variable ω can visit the fire-sale region with non-trivial probability as $\chi \to 0$.

Numerical illustration. We now provide a numerical illustration of how the equilibrium depends on χ. As the analysis above suggests, the most convenient way to investigate the equilibrium is by using the transformed state variable $\omega := \eta/\chi$, whose ergodic set is $[0,1]$. We thus solve ODE (F.19) for $\hat{q}$ as a function of ω, then compute $\hat{\kappa}$ via (F.21), and finally compute the other equilibrium objects like the drift μ_ω and σ_ω via (F.26)-(F.27). The results are depicted in Figure F.1.

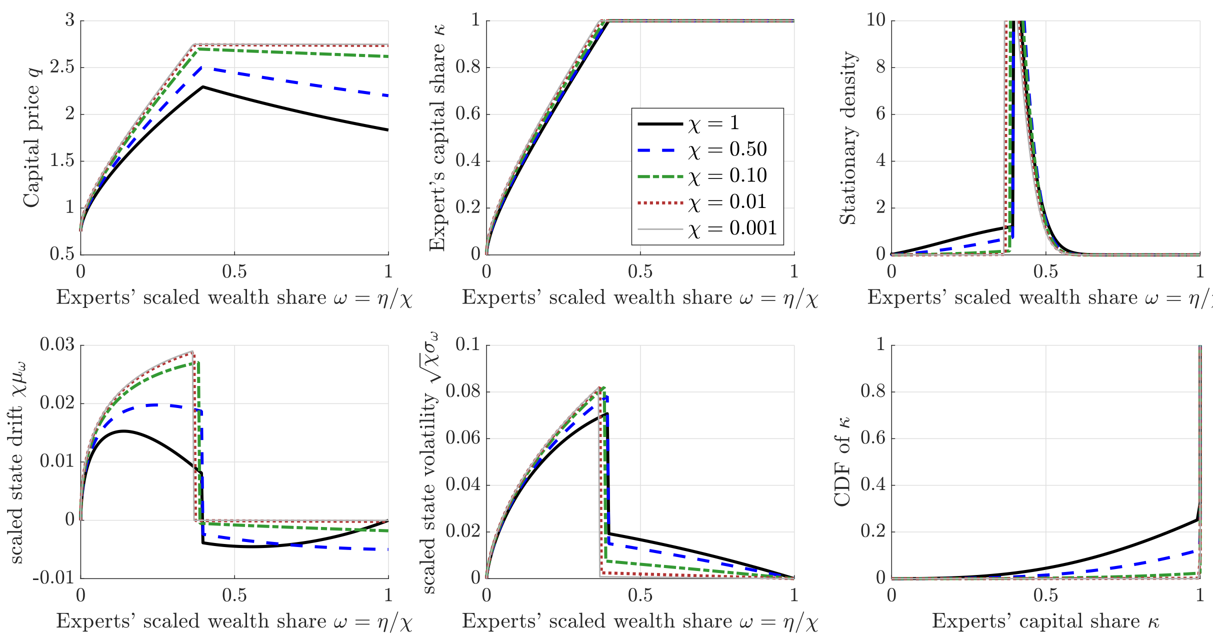

Figure F.1: Equilibria as equity-issuance frictions improve, $\chi \to 0$. Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.04$. In all cases, we use the boundary condition $\kappa(0) = 0$.

It is noteworthy that, indeed as suggested by Proposition F.1-F.2, the capital price and expert capital share stabilize as $\chi \to 0$ to functions which have a non-trivial fire-sale region (first and second panels). This is the sense in which the equilibrium looks similar for any χ. That being said, the probability of fire sales shrinks as χ shrinks, and eventually vanishes as $\chi \to 0$ (third and sixth panels). This intuitive result is not obvious analytically, as discussed in Remark F.1 above. In particular, the fourth and fifth panels plot the scaled drift and diffusion, $\chi \mu_\omega$ and $\sqrt{\chi}\sigma_\omega$, respectively, which stabilize as $\chi \to 0$, implying that the transformed state dynamics become explosive in the limit. It turns out that the explosiveness of the drift dominates, pushing the economy very quickly out of the fire-sale region, such that the stationary distribution does not admit any fire sales as $\chi \to 0$.

G Discrete-time model

The following discrete-time model is exactly analogous to our continuous-time model. The setup and exposition is based on the discrete-time model in the online appendix of Khorrami and Mendo (2024), but the analysis diverges when we study the Markovian equilibrium with expert wealth share η as the sole state variable (whereas that paper studies a non-Markovian equilibrium). We model each decision on a time-step of Δ, and we will also consider limits as $\Delta \to 0$ to compare to our baseline model.

Technology. For simplicity, we assume that aggregate capital K is fixed, i.e., there is no exogenous growth or fundamental uncertainty ($g = \sigma = 0$).

Individual agent problem. An individual can hold two assets, riskless bonds b_t and capital k_t, and decides consumption c_t. The individual net worth, just before consuming, is $n_t = b_t + q_t k_t$, where q_t is the market price of capital. The one-period return on bonds is $R_t^f = 1 + r_t \Delta$, and the return-on-capital is $R_{t+\Delta}^k := \frac{a\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t}$, where a is the agent's productivity per unit of time while holding capital. Then, the agent's dynamic budget constraint is[28]

$$n_{t+\Delta} = q_t k_t (R_{t+\Delta}^k - R_t^f) + (n_t - c_t) R_t^f. \tag{G.1}$$

Each agent takes q_t, R_t^f, and $R_{t+\Delta}^k$ as given and chooses (c, k, n) to maximize

$$\mathbb{E}\left[\sum_{i=0}^{\infty} \left(\frac{1}{1+\rho\Delta}\right)^i \log(c_{i\Delta})\right], \tag{G.2}$$

subject to (G.1), subject to the no-shorting constraint $k_t \geq 0$, and subject to the solvency constraint $n_t \geq 0$. We use the expectation operator to allow for the possibility that sunspot shocks can drive fluctuations.

[28]To derive (G.1), proceed as follows. First, note that the bond market account next period, before adjusting the portfolio of bonds and capital, will have value $b'_{t+\Delta} = R_f^f (b_t - c_t) + a k_t \Delta$—that is, after consumption expenditures are made, the residual earns the interest rate, and the cash flows from holding capital are also added at the end of the period. Second, the capital holdings k_t will have value $q_{t+\Delta} k_t$ next period. Adding these two quantities must equal tomorrow's net worth $n_{t+\Delta}$. Hence, $n_{t+\Delta} = R_f^f (b_t - c_t) + a k_t \Delta + q_{t+\Delta} k_t$. Using the definition $n_t = b_t + q_t k_t$ gives the result (G.1).

The first-order optimality conditions are the standard Euler equations

$$1 = \frac{1}{1+\rho\Delta} R_t^f \mathbb{E}_t\Big[\frac{c_t}{c_{t+\Delta}}\Big] \tag{G.3}$$

$$0 \geq \frac{1}{1+\rho\Delta} \mathbb{E}_t\Big[\frac{c_t}{c_{t+\Delta}}(R_{t+\Delta}^k - R_t^f)\Big], \tag{G.4}$$

where (G.4) holds with equality when $k_t > 0$ is chosen.

In addition, it is straightforward to show that optimal consumption satisfies the standard log utility formula[29]

$$c_t = \frac{\rho\Delta}{1+\rho\Delta} n_t. \tag{G.5}$$

Using this fact, plus the budget constraint (G.1) in (G.3)-(G.4), we obtain

$$1 = \frac{1}{1+\rho\Delta} R_t^f \mathbb{E}_t\left[\frac{1}{\theta_t(R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1}R_t^f}\right] \tag{G.6}$$

$$0 \geq \frac{1}{1+\rho\Delta} \mathbb{E}_t\left[\frac{R_{t+\Delta}^k - R_t^f}{\theta_t(R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1}R_t^f}\right], \quad \text{with equality if } \theta_t > 0 \tag{G.7}$$

where $\theta_t := \frac{q_t k_t}{n_t}$ is the share of wealth allocated to capital. At this point, one can prove that (G.6) holds automatically if (G.7) holds.[30] Therefore, we can drop the bond Euler equation (G.6) from the remainder of the analysis, i.e., (G.5) and (G.7) fully characterize the agent's optimal choices.

Aggregation and equilibrium conditions. As in the main text, we assume there are two types of agents: experts have productivity a_e and discount rate ρ_e, while households have productivity $a_h < a_e$ and discount rate $\rho_h \leq \rho_e$. Clearly, then, experts have a higher return-on-capital than households: $R_{e,t+\Delta}^k > R_{h,t+\Delta}^k$.

We now aggregate. The market clearing condition for goods, capital, and bonds are

[29]This can be showed by writing out the Bellman equation and guessing-and-verifying that the value function takes the form $v_t = (1-\beta)^{-1}\log(n_t) + f(\Omega_t)$ for $\beta = (1+\rho\Delta)^{-1}$ and some function f that only depends on aggregate states Ω_t. Then, the envelope condition says $c_t^{-1} = \frac{\partial}{\partial n} v_t = (1-\beta)^{-1} n_t^{-1}$, which is the consumption formula.

[30]Indeed, if $\theta_t = 0$ it is obvious that (G.6) holds. If $\theta_t > 0$, then (G.7) holds with equality, so we then have

$$0 = \mathbb{E}_t\Big[\frac{\theta_t(R_{t+\Delta}^k - R_t^f)}{\theta_t(R_{t+\Delta}^k - R_t^f) + (1+\rho\Delta)^{-1}R_t^f}\Big]$$

Adding this expression to equation (G.6), we obtain the identity $1 = 1$.

given by, respectively,

$$c_{e,t} + c_{h,t} = (a_e k_{e,t} + a_h k_{h,t})\Delta \tag{G.8}$$

$$k_{e,t} + k_{h,t} = K \tag{G.9}$$

$$b_{e,t} + b_{h,t} = c_{e,t} + c_{h,t}. \tag{G.10}$$

Equation (G.10) says that bondholdings just after consuming (which is $b_t - c_t$) sum to the zero net supply. By combining (G.10) with the individual net worth definition $n_t = b_t + q_t k_t$, we obtain an alternative statement of bond market clearing that we will use:

$$n_{e,t} + n_{h,t} = q_t K + c_{e,t} + c_{h,t}. \tag{G.11}$$

Definition 5. An *equilibrium* is a collection of stochastic processes for allocations $(k_{j,t\Delta}, n_{j,t\Delta}, c_{j,t\Delta})_{t=0}^{\infty}$ for $j \in \{e, h\}$ with $k_{e,0}$ and $k_{h,0}$ given, and for prices $(q_{t\Delta}, R^f_{t\Delta})_{t=0}^{\infty}$ such that (i) given prices, allocations solve each agent type's problem, and (ii) markets clear.

G.1 Equilibrium characterization

We have already characterized optimal decisions and market clearing conditions. In particular, a collection of stochastic processes for allocations and prices constitute an equilibrium if they satisfy (G.1), (G.5), and (G.7) for each agent type (experts and households), along with equations (G.8), (G.9), and (G.11) at the aggregate level.

We further tighten this characterization and reduce it to four stochastic processes satisfying a set of conditions, exactly as in our continuous-time model. First, to keep track of the distribution of wealth and capital, let $\eta_t := (1+\rho_e\Delta)^{-1} n_{e,t}/q_t K$ and $\kappa_t := k_{e,t}/K$ denote expert's wealth and capital shares.[31] Whereas κ_t is a "jumpy" variable because it is linked to agent's capital choices, η_t is a "state" variable because it is determined via agent's slow-moving wealths. Using the budget constraint (G.1), we can obtain the dynamics of η_t as

$$\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta}\left(\frac{\kappa_t(R^k_{e,t+\Delta} - R^f_t) + \eta_t R^f_t}{q_{t+\Delta}/q_t}\right). \tag{G.12}$$

Next, we aggregate the consumption decisions across these two types. To do this, plug the consumption rules from (G.5) into the goods and bond market clearing conditions

[31] Note that the wealth share is defined just after consumption choices are made, i.e., $\eta_t = (n_{e,t} - c_{e,t})/(n_{e,t} + n_{h,t} - c_{e,t} - c_{h,t})$ is the definition we are using.

(G.8) and (G.11), and combine the results to obtain

$$q_t \bar{\rho}(\eta_t) = \kappa_t a_e + (1 - \kappa_t) a_h, \tag{G.13}$$

where $\bar{\rho}(\eta) := \eta \rho_e + (1 - \eta)\rho_h$ is a wealth-weighted average discount rate. Identical to our continuous-time model, equation (G.13) is a *price-output relation* that links asset values q_t to the efficiency of the capital distribution κ_t. Finally, we aggregate the Euler equations (G.7) within the two types using the fact that experts will always be on the margin (i.e., since $R^k_{e,t+\Delta} > R^k_{h,t+\Delta}$, we have $k_{e,t} > 0$ at all times). We also use the fact that $\theta_{e,t} = \frac{q_t k_{e,t}}{n_{e,t}} = \frac{1}{1+\rho_e \Delta} \frac{\kappa_t}{\eta_t}$ and $\theta_{h,t} = \frac{q_t k_{h,t}}{n_{h,t}} = \frac{1}{1+\rho_h \Delta} \frac{1-\kappa_t}{1-\eta_t}$ to write the results in a more convenient way. The results are

$$0 = \mathbb{E}_t \left[\frac{q_{t+\Delta} + a_e \Delta - R^f_t q_t}{\frac{\kappa_t}{\eta_t} \left(q_{t+\Delta} + a_e \Delta - R^f_t q_t \right) + R^f_t q_t} \right] \tag{G.14}$$

$$0 \geq \mathbb{E}_t \left[\frac{q_{t+\Delta} + a_h \Delta - R^f_t q_t}{\frac{1-\kappa_t}{1-\eta_t} \left(q_{t+\Delta} + a_h \Delta - R^f_t q_t \right) + R^f_t q_t} \right] \tag{G.15}$$

where the latter holds as an equality when households hold capital, i.e., when $\kappa_t < 1$.

Thus, an equilibrium is fully characterized by the collection of stochastic processes $(\eta_{t\Delta}, \kappa_{t\Delta}, q_{t\Delta}, R^f_{t\Delta})_{t=0}^{\infty}$, with $\eta_0 = k_{e,0}/K$ given, such that the two optimality conditions (G.14)-(G.15) hold; the price-output relation (G.13) holds; and the law of motion for η_t is given by (G.12). We state this characterization as a lemma.

Lemma G.1. *Given $\eta_0 \in (0, 1)$, consider stochastic processes $\{\eta_{t\Delta}, q_{t\Delta}, \kappa_{t\Delta}, R^f_{t\Delta}\}_{t=0}^{\infty}$ such that η_t evolution is described by (G.12). If $\eta_t \in [0, 1]$, $\kappa_t \in [0, 1]$, and equations (G.13), (G.14), and (G.15) hold for all $t \geq 0$, then $\{\eta_{t\Delta}, q_{t\Delta}, \kappa_{t\Delta}, R^f_{t\Delta}\}_{t=0}^{\infty}$ corresponds to an equilibrium.*

Among all possible equilibria, we focus on equilibria which are Markovian in η_t. That is, we restrict q_t, κ_t, and R^f_t (hence $r_t = (R^f_t - 1)/\Delta$) to be solely functions of η_t. We now analyze the two types of equilibria: fundamental and sunspot.

G.2 Fundamental equilibrium

A *fundamental equilibrium* has $\kappa_t = 1$ for all periods. In such an equilibrium, (G.13) says that the capital price should be

$$q_t = \frac{a_e}{\bar{\rho}(\eta_t)}, \quad \text{if} \quad \kappa_t = 1. \tag{G.16}$$

Substituting this result into the state dynamics (G.12), we have

$$\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta}\Big[1+\bar{\rho}(\eta_{t+\Delta}) - \frac{\bar{\rho}(\eta_{t+\Delta})}{\bar{\rho}(\eta_t)}(1-\eta_t)R_t^f\Big], \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{G.17}$$

As the only $(t+\Delta)$-measurable object in (G.17), $\eta_{t+\Delta}$ evolves deterministically in a fundamental equilibrium. Because q_t is solely a function of η_t in (G.16), $q_{t+\Delta}$ is also known as of time t. As a result, experts' return-on-capital must coincide with the riskless rate, i.e., $R_t^f = \frac{a_e\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t}$, or

$$R_t^f = \bar{\rho}(\eta_t) + \frac{\bar{\rho}(\eta_t)}{\bar{\rho}(\eta_{t+\Delta})}, \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{G.18}$$

Combining (G.17) and (G.18), we obtain the solved dynamics

$$\eta_{t+\Delta} = \frac{\eta_t(1+\rho_e\Delta)^{-1}}{\eta_t(1+\rho_e\Delta)^{-1} + (1-\eta_t)(1+\rho_h\Delta)^{-1}}, \quad \text{if} \quad \kappa_t = \kappa_{t+\Delta} = 1. \tag{G.19}$$

Thus, expert's wealth share asymptotically tends toward zero. Intuitively, they earn zero excess capital returns and consume at a higher rate than households.

G.3 Sunspot equilibrium

A *sunspot equilibrium* has $\kappa_t < 1$ for some t. We proceed with a simple binomial tree example to show that sunspot equilibria exist, although more complicated information structures are also likely possible. We conjecture an equilibrium with

$$q_{t+\Delta} = \begin{cases} u_t q_t, & \text{with probability } 1-p_t; \\ d_t q_t, & \text{with probability } p_t. \end{cases} \tag{G.20}$$

The shock, which is whether the price goes "up" or "down", is a pure sunspot shock. The "up" and "down" returns u_t and $d_t \in (0, u_t)$ may be state dependent, as may the probability of a price drop p_t. In other words, (u_t, d_t, p_t) will be functions of η_t, as will the interest rate r_t.

To start, we may solve for the optimal portfolios explicitly in this binomial environment. Using (G.12) and (G.20) in the expert Euler equation (G.14), we have

$$\frac{\kappa_t}{\eta_t} = -R_t^f \frac{(1-p_t)u_t + p_t d_t + \frac{a_e\Delta}{q_t} - R_t^f}{(u_t + \frac{a_e\Delta}{q_t} - R_t^f)(d_t + \frac{a_e\Delta}{q_t} - R_t^f)}. \tag{G.21}$$

Doing the same for the household Euler equation (G.15), we have

$$\frac{1-\kappa_t}{1-\eta_t} = -R_t^f \min\left(0, \frac{(1-p_t)u_t + p_t d_t + \frac{a_h\Delta}{q_t} - R_t^f}{(u_t + \frac{a_h\Delta}{q_t} - R_t^f)(d_t + \frac{a_h\Delta}{q_t} - R_t^f)}\right). \tag{G.22}$$

Next, note that the price-output relation (G.13) and state dynamics (G.12) are unchanged by the binomial setup, and we repeat them here for convenience:

$$\bar{\rho}(\eta_t) = \frac{\kappa_t a_e + (1-\kappa_t)a_h}{q_t} \tag{G.23}$$

$$\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta}\frac{\kappa_t(\frac{a_e\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t} - R_t^f) + \eta_t R_t^f}{q_{t+\Delta}/q_t}. \tag{G.24}$$

As mentioned in Lemma G.1, to find an equilibrium we only need to check that we can pick (u_t, d_t, p_t) to satisfy (G.21)-(G.24) at every point in the state space and that the resulting equilibrium dynamics are "stable" in the sense that they do not cause the dynamical system to "exit the feasible region." To this end, we immediately note that $\eta_t \in (0,1)$ on any equilibrium path, which can be verified by checking the state dynamics (G.24).[32] Consequently, by focusing on Markovian equilibria in η, we automatically ensure "stochastic stability." But how do we solve for such a Markovian equilibrium?

In the Markovian case, we need only check that (G.21)-(G.24) hold for appropriate functions $q(\cdot)$, $\kappa(\cdot)$, $R^f(\cdot)$, $u(\cdot)$, $d(\cdot)$, $p(\cdot)$ taking η as an input. Writing the first three

[32] Examine the state dynamics (G.24) in the down state and substitute (G.21) to obtain

$$d_t\frac{\eta_{t+\Delta}^d}{\eta_t} = \frac{1}{1+\rho_e\Delta}R_t^f\Big(1 - \frac{(1-p_t)u_t + p_t d_t + \frac{a_e\Delta}{q_t} - R_t^f}{u_t + \frac{a_e\Delta}{q_t} - R_t^f}\Big) > 0.$$

Similarly, mirroring (G.24), the symmetric condition for household's net worth share dynamics is

$$1-\eta_{t+\Delta} = \frac{1}{1+\rho_e\Delta}\frac{(1-\kappa_t)(\frac{a_h\Delta}{q_t} + \frac{q_{t+\Delta}}{q_t} - R_t^f) + (1-\eta_t)R_t^f}{q_{t+\Delta}/q_t}$$

Examining this condition in the up state and substituting (G.22), we obtain

$$u_t\frac{1-\eta_{t+\Delta}^u}{1-\eta_t} = \frac{1}{1+\rho_h\Delta}R_t^f\Big(1 - \min\Big(0, \frac{(1-p_t)u_t + p_t d_t + \frac{a_h\Delta}{q_t} - R_t^f}{d_t + \frac{a_h\Delta}{q_t} - R_t^f}\Big)\Big) > 0.$$

Thus, the requirement to keep $\eta_t \in (0,1)$ is automatically satisfied.

equations out explicitly, we need to find functions such that

$$\frac{\kappa(\eta)}{\eta} = -R^f(\eta)\frac{(1-p(\eta))u(\eta)+p(\eta)d(\eta)+\frac{a_e\Delta}{q(\eta)}-R^f(\eta)}{\left(u(\eta)+\frac{a_e\Delta}{q(\eta)}-R^f(\eta)\right)\left(d(\eta)+\frac{a_e\Delta}{q(\eta)}-R^f(\eta)\right)} \tag{G.25}$$

$$\frac{1-\kappa(\eta)}{1-\eta} = -R^f(\eta)\min\left(\frac{(1-p(\eta))u(\eta)+p(\eta)d(\eta)+\frac{a_h\Delta}{q(\eta)}-R^f(\eta)}{\left(u(\eta)+\frac{a_h\Delta}{q(\eta)}-R^f(\eta)\right)\left(d(\eta)+\frac{a_h\Delta}{q(\eta)}-R^f(\eta)\right)}\right) \tag{G.26}$$

$$\bar{\rho}(\eta) = \frac{\kappa(\eta)a_e+(1-\kappa(\eta))a_h}{q(\eta)} \tag{G.27}$$

In addition, the functions $u(\cdot)$ and $d(\cdot)$ must be "consistent" with the function $q(\cdot)$, in the sense that

$$u(\eta)q(\eta) = q\left(\frac{1}{1+\rho_e\Delta}\frac{\kappa(\eta)\left(\frac{a_e\Delta}{q(\eta)}+u(\eta)-R^f(\eta)\right)+\eta R^f(\eta)}{u(\eta)}\right) \tag{G.28}$$

$$d(\eta)q(\eta) = q\left(\frac{1}{1+\rho_e\Delta}\frac{\kappa(\eta)\left(\frac{a_e\Delta}{q(\eta)}+d(\eta)-R^f(\eta)\right)+\eta R^f(\eta)}{d(\eta)}\right) \tag{G.29}$$

Equations (G.28)-(G.29) come from ensuring that the conjectured pricing function $q(\cdot)$, evaluated at next period's η given by (G.24), equals the value of q defined by the functions u and d. We call these the "Markov consistency conditions" for short.

To continue, we will specialize below to particular choices of u, d, and p. First, we will pursue a construction which serves as an approximation to Brownian motion, allowing us to show convergence to our BSE as $\Delta \to 0$. Second, we demonstrate a construction that approximates Poisson shocks and converges to our PSE of Section 4 as $\Delta \to 0$. Thus, the binomial setup considered here can unify all of our sunspot information structures.

As in the baseline setting, we conjecture (but do not prove here) the existence of a threshold $\eta^* \in (0,1)$ such that $\kappa(\eta) = 1$ if and only if $\eta \geq \eta^*$. Furthermore, there will be another threshold $\tilde{\eta}^*$, to be defined below, at which $\kappa(\tilde{\eta}^*)$ is sufficiently close to 1, such that positive shocks will just take the economy to the border. With these thresholds, we split up our analysis into three regions:

$$\begin{aligned}
&\text{(fire-sale region)} && \mathcal{D}^\circ := \{\eta \in (0,\tilde{\eta}^*)\}\\
&\text{(close-to-efficient region)} && \tilde{\mathcal{D}}_1 := \{\eta \in [\tilde{\eta}^*,\eta^*)\}\\
&\text{(efficient region)} && \mathcal{D}_1 := \{\eta \in [\eta^*,1)\}
\end{aligned}$$

Because endogenous changes in variables will be proportional to the time-step Δ, the "close-to-efficient" region $\tilde{\mathcal{D}}_1$ will collapse to a null-set as $\Delta \to 0$. Furthermore, the efficient region $\mathcal{D}_1$ involves a straightforward analysis in which prices can potentially jump down into $\mathcal{D}^\circ$ with some probability, but where this probability vanishes as $\Delta \to 0$ (i.e., going to continuous time, the capital price becomes continuous outside of Poisson jumps, and so the probability of jumping from efficiency down into the fire-sale region is of "order dt"). And so to streamline the discussion, we simply analyze $\mathcal{D}^\circ$ below.

G.4 Brownian approximation

In the fire-sale region $\mathcal{D}^\circ$, we construct a sunspot equilibrium by explicitly specifying (u, d, p) to take a form that approximates Brownian motion in the $\Delta \to 0$ limit. In particular, we set

$$u = 1 + v\sqrt{\Delta} \tag{G.30}$$

$$d = 1 - v\sqrt{\Delta} \tag{G.31}$$

$$p = \frac{v - m\sqrt{\Delta}}{2v}, \tag{G.32}$$

for some endogenous functions m and v. Note that $p \in (0,1)$ requires $m\sqrt{\Delta} \in (-v, v)$. Of course, we also require $v \leq 1/\sqrt{\Delta}$, so that the downward jump is not greater than 100%. These constraints on m and v become arbitrarily loose as $\Delta \to 0$.

One can verify that (G.30)-(G.32) imply that

$$\mathbb{E}_t[\frac{q_{t+\Delta} - q_t}{q_t}] = m(\eta_t)\Delta \quad \text{and} \quad \mathbb{E}_t[(\frac{q_{t+\Delta} - q_t}{q_t})^2] = v(\eta_t)^2\Delta.$$

Thus, the interpretation of m and v are as the drift and instantaneous volatility of percentage price changes, analogous to the μ_q and σ_q in continuous time. In fact, one can observe that $(\frac{q_{t+\Delta}-q_t}{q_t})^2 = v(\eta_t)^2\Delta$ even without the expectation operator, analogous to the deterministic nature of quadratic variation. Finally, notice that any higher moments of price changes are of order $o(\Delta)$.

Similarly, substituting the specification (G.30)-(G.32) into (G.24), one can verify that the state dynamics converge as $\Delta \to 0$ to the continuous-time model. Indeed, examine the conditional mean and second moment of $\eta_{t+\Delta} - \eta_t$:

$$\mathbb{E}_t[\eta_{t+\Delta} - \eta_t] = \Big(\kappa_t \frac{a_e}{q_t} - \eta_t\rho_e + (\kappa_t - \eta_t)(m_t - r_t - v_t^2)\Big)\Delta + o(\Delta) \tag{G.33}$$

$$\mathbb{E}_t[(\eta_{t+\Delta} - \eta_t)^2] = (\kappa_t - \eta_t)^2 v_t^2 \Delta + o(\Delta), \tag{G.34}$$

where $m_t = m(\eta_t)$ and $v_t = v(\eta_t)$. Dividing by Δ and taking $\Delta \to 0$, it becomes clear that these moments coincide with those of the continuous-time model, so long as $(q_t, \kappa_t, v_t, m_t, r_t)$ also converge to their continuous-time versions. For reference, let us re-write the dynamics of η in a different way:

$$\eta_{t+\Delta} - \eta_t = \mathbb{E}_t[\eta_{t+\Delta} - \eta_t] + (\kappa_t - \eta_t)\Big(\frac{q_{t+\Delta} - q_t}{q_t} - (m_t - v_t^2)\Delta\Big) + o(\Delta). \quad \text{(G.35)}$$

Now, we determine what m and v must be to satisfy agents' optimality conditions. In this Brownian approximation, the expert and household Euler equations (G.25)-(G.26) become

$$\frac{\kappa}{\eta} = (1 + r\Delta)\frac{\frac{a_e}{q} + m - r}{v^2 - (\frac{a_e}{q} - r)^2\Delta} \quad \text{(G.36)}$$

$$\frac{1-\kappa}{1-\eta} = (1 + r\Delta)\max\Big\{0, \frac{\frac{a_h}{q} + m - r}{v^2 - (\frac{a_h}{q} - r)^2\Delta}\Big\}. \quad \text{(G.37)}$$

First we use the expert Euler equation to solve for v^2:

$$v^2 = (1 + r\Delta)\Big[\frac{a_e}{q} + m - r\Big]\frac{\eta}{\kappa} + (\frac{a_e}{q} - r)^2\Delta.$$

Then, we use the household Euler equation, when $\kappa_t < 1$, to also solve for v^2:

$$v^2 = (1 + r\Delta)\Big[\frac{a_h}{q} + m - r\Big]\frac{1-\eta}{1-\kappa} + (\frac{a_h}{q} - r)^2\Delta.$$

Setting these expressions equal gives an equation for m, which satisfies

$$m = r + \frac{(1-\kappa)\eta}{\kappa - \eta}\frac{a_e}{q} - \frac{\kappa(1-\eta)}{\kappa - \eta}\frac{a_h}{q} + \frac{\kappa(1-\kappa)\big[(\frac{a_e}{q} - r)^2 - (\frac{a_h}{q} - r)^2\big]}{(1 + r\Delta)(\kappa - \eta)}\Delta. \quad \text{(G.38)}$$

Substituting back into the equations for v^2, we solve for

$$v^2 = (1 + r\Delta)\frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q} + \frac{\kappa(1-\eta)(\frac{a_e}{q} - r)^2 - \eta(1-\kappa)(\frac{a_h}{q} - r)^2}{\kappa - \eta}\Delta. \quad \text{(G.39)}$$

Note that these equations, in the $\Delta \to 0$ limit, are identical to the continuous-time versions (when there is zero fundamental risk and zero growth). Indeed, equation (G.39) says

$$v^2 = \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q} + O(\Delta). \tag{G.40}$$

Next, by doing some algebra on (G.38), it reads

$$m = r - \bar{\rho}(\eta) + \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)v^2 + O(\Delta). \tag{G.41}$$

Consequently, m and v are indeed the discrete-time counterparts to μ_q and σ_q. Until now, the argument has not used the fact that the equilibrium is Markovian.

To fully solve the model, we need to check the "Markov consistency conditions" in (G.28)-(G.29) which make sure that the function $q(\cdot)$ is consistent with the state dynamics. Given we are planning to take $\Delta \to 0$, we can check these in an easier way for the Brownian case. Indeed, under the Markov assumption, we may use a second-order Taylor expansion to obtain

$$q_{t+\Delta} - q_t = q'(\eta_t) \times \Big(\eta_{t+\Delta} - \eta_t\Big) + \frac{1}{2}q''(\eta_t) \times \Big(\eta_{t+\Delta} - \eta_t\Big)^2 + o\Big(\eta_{t+\Delta} - \eta_t\Big)^2$$

Then, using (G.35), followed by $(\frac{q_{t+\Delta}-q_t}{q_t})^2 = v_t^2\Delta$, we have

$$\begin{aligned}
q_{t+\Delta} - q_t &= q'(\eta_t)(\eta_{t+\Delta} - \eta_t) + \frac{1}{2}q''(\eta_t)(\eta_{t+\Delta} - \eta_t)^2 + o(\Delta) \\
&= q'(\eta_t)(\kappa_t - \eta_t)\frac{q_{t+\Delta} - q_t}{q_t} + q'(\eta_t)\Big(\mathbb{E}_t[\eta_{t+\Delta} - \eta_t] - (\kappa_t - \eta_t)(m_t - v_t^2)\Delta\Big) \\
&\quad + \frac{1}{2}q''(\eta_t)(\kappa_t - \eta_t)v_t^2\Delta + o(\Delta)
\end{aligned}$$

Noticing that $q_{t+\Delta} - q_t$ appears on the left- and right-hand-side, we may solve this equation to obtain

$$\frac{q_{t+\Delta} - q_t}{\sqrt{\Delta}q_t} = \frac{\frac{1}{\sqrt{\Delta}}\big[\frac{q'(\eta_t)}{q_t}\big(\mathbb{E}_t[\eta_{t+\Delta} - \eta_t] - (\kappa_t - \eta_t)(m_t - v_t^2)\Delta\big) + \frac{1}{2}\frac{q''(\eta_t)}{q_t}(\kappa_t - \eta_t)v_t^2\Delta + o(\Delta)\big]}{1 - \frac{q'(\eta_t)}{q_t}(\kappa_t - \eta_t)}$$

From the binomial approximation, the left-hand-side is $\pm v_t = O(1)$, assuming volatility is non-zero. But the numerator of the right-hand-side is of order $O(\sqrt{\Delta})$. Consequently, the denominator of the right-hand-side must also be $O(\sqrt{\Delta})$. Thus, we obtain:

$$1 - \frac{q'(\eta)}{q(\eta)}(\kappa - \eta) = O(\sqrt{\Delta}) \tag{G.42}$$

As $\Delta \to 0$, this converges exactly to the BSE differential equation (21). This proves that $q(\cdot)$ is consistent with the state dynamics for η, since (G.42) holds at every point in the fire-sale region.

G.5 Poisson jump approximation

Now, we specialize to a discretization (u, d, p) in the fire-sale region $\mathcal{D}^\circ$ that approximates our Poisson uncertainty from Section 4. In particular, we set

$$u = 1 + m\Delta \tag{G.43}$$

$$d = 1 - z \tag{G.44}$$

$$p = \lambda\Delta, \tag{G.45}$$

for some endogenous functions $m > 0$, $z \in (0, 1)$, and for $\Delta < 1/\lambda$ (so that $p < 1$ is a well-defined probability). The capital price either drifts up slightly, with probability $1 - \lambda\Delta$, or it jumps down by percentage z, with probability $\lambda\Delta$. The rate of a down-jump is exactly λ per unit of time, exactly as in the Poisson model in Section 4. Consequently, the wealth share dynamics in (G.24) necessarily converge to the desired Poisson dynamics, so long as $z \sim \zeta_q$ and $m \sim \mu_q$ as $\Delta \to 0$.

The algebra to verify convergence as $\Delta \to 0$ to the Poisson Sunspot Equilibrium is as follows. The expert and household Euler equations (G.25)-(G.26) become, when $\kappa_t < 1$,

$$\frac{\kappa}{\eta} = -(1 + r\Delta)\frac{\frac{a_e}{q} + m - r - \lambda z - m\lambda\Delta}{(\frac{a_e}{q} + m - r)((\frac{a_e}{q} - r)\Delta - z)} \tag{G.46}$$

$$\frac{1-\kappa}{1-\eta} = -(1 + r\Delta)\frac{\frac{a_h}{q} + m - r - \lambda z - m\lambda\Delta}{(\frac{a_h}{q} + m - r)((\frac{a_h}{q} - r)\Delta - z)}. \tag{G.47}$$

Rearrange these two equations and group terms to get

$$\Big(\frac{a_e}{q} + m - r\Big)\Big[1 - \frac{\kappa}{\eta}z + \Big(\frac{\kappa}{\eta}(\frac{a_e}{q} - r) + r\Big)\Delta\Big] = (1 + r\Delta)\Big(\lambda z + m\lambda\Delta\Big)$$

$$\Big(\frac{a_h}{q} + m - r\Big)\Big[1 - \frac{1-\kappa}{1-\eta}z + \Big(\frac{1-\kappa}{1-\eta}(\frac{a_h}{q} - r) + r\Big)\Delta\Big] = (1 + r\Delta)\Big(\lambda z + m\lambda\Delta\Big)$$

Now, we guess-and-verify that $z = O(1)$ and $m = O(1)$ as $\Delta \to 0$. In that case, the previous two equations can be written

$$\frac{a_e}{q} + m - r = \frac{\lambda z}{1 - \frac{\kappa}{\eta} z} + O(\Delta) \tag{G.48}$$

$$\frac{a_h}{q} + m - r = \frac{\lambda z}{1 - \frac{1-\kappa}{1-\eta} z} + O(\Delta) \tag{G.49}$$

To complete the argument, take (i) the difference between (G.48)-(G.49) and (ii) a $(\kappa, 1 - \kappa)$ weighted sum of (G.48)-(G.49) to obtain the following two equations:

$$\frac{a_e - a_h}{q} = \frac{\kappa - \eta}{\eta(1-\eta)} \frac{\lambda z^2}{(1 - \frac{\kappa}{\eta} z)(1 - \frac{1-\kappa}{1-\eta} z)} + O(\Delta) \tag{G.50}$$

$$r = \frac{\kappa a_e + (1-\kappa) a_h}{q} + m - \lambda z \Big(\frac{\kappa}{1 - \frac{\kappa}{\eta} z} + \frac{1-\kappa}{1 - \frac{1-\kappa}{1-\eta} z} \Big) + O(\Delta) \tag{G.51}$$

As $\Delta \to 0$, these equations converge to the risk-balance equation (when $\kappa < 1$) in Section 4 and the riskless rate equation in Appendix E. Consequently, in a Markov equilibrium, z converges to the downward jump-size ζ_q and m converges to the drift μ_q as $\Delta \to 0$. Up to now, the argument has not relied on the fact that the equilibrium is Markovian in η.

To fully verify the limit, we must verify the "Markov consistency conditions" in (G.28)-(G.29) converge to their Poisson Sunspot Equilibrium counterpart in Section 4. First, since $u \to 1$ as $\Delta \to 0$, equation (G.28) automatically holds in the limit. Second, condition (G.29), in the limit $\Delta \to 0$, says

$$(1 - z(\eta)) q(\eta) = q\Big(\frac{-\kappa(\eta) z(\eta) + \eta}{1 - z(\eta)} \Big) \tag{G.52}$$

Equation (G.52) is exactly identical to the PSE equation (29).[33]

[33] To see the equivalence, start from equation (29) and use the definition of $\hat{\eta} := \eta - \zeta_\eta = \eta - (\kappa - \eta) \frac{\zeta_q}{1-\zeta_q}$ to write

$$\zeta_q = -\frac{q(\hat{\eta}) - q(\eta)}{q(\eta)} = -\frac{q(\eta - (\kappa - \eta)\frac{\zeta_q}{1-\zeta_q}) - q(\eta)}{q(\eta)}$$

Doing some algebra and rearranging terms, we have

$$1 - \zeta_q = q\Big(\frac{-\kappa \zeta_q + \eta}{1 - \zeta_q} \Big)$$

Given the conjecture that $z \sim \zeta_q$ as $\Delta \to 0$, this condition is asymptotically identical to condition (G.52).